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Non-Abelian vortices in $\mathrm{SO}(N)$ and $\mathrm{USp}(N)$ gauge theories

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Abstract: Non-Abelian BPS vortices in $\mathrm{SO}(N)\times\mathrm{U}(1)$ and $\mathrm{USp}(2N)\times\mathrm{U}(1)$ gauge theories are constructed in maximally color-flavor locked vacua. We study in detail their moduli and transformation properties under the exact symmetry of the system. Our results generalize non-trivially those found earlier in supersymmetric $\mathrm{U}(N)$ gauge theories. The structure of the moduli spaces turns out in fact to be considerably richer here than what was found in the $\mathrm{U}(N)$ theories. We find that vortices are generally of the semi-local type, with power-like tails of profile functions.

Keywords: Supersymmetric gauge theory, Spontaneous Symmetry Breaking, Duality in Gauge Field Theories, Solitons Monopoles and Instantons

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Contents

1 Introduction

Solitons play an important role in a wide range of physics, from condensed-matter and fluid dynamics to cosmology and particle physics [1]–[6]. A quiet revolution in our understanding of soliton vortices has been taking place in the last few years, in the context of supersymmetric gauge theories, triggered by the discovery of *vortices of non-Abelian type* [7, 8]. The latter represent continuous families of vortex solutions carrying various moduli corresponding to the internal orientations (related to the exact flavor symmetry of the system) as well as other zero-modes. It is possible that such non-Abelian vortices are a key to unravel the mystery of confinement in Quantum Chromodynamics (QCD).

Motivated by this, together with other physics interests, many related questions have been investigated systematically by several groups [9]–[41]. The moduli-matrix formalism was introduced in refs. [9]–[12] in order to exhaust all possible moduli. The moduli and transformation properties of the moduli in the cases of composite vortices (higher-winding vortices) [13–16] and semi-local vortices [17, 18], have been studied. A new type of (Seiberg-like) duality was found among pairs of models having related vortex moduli, sharing a common sigma-model-lump limit [18]. Systems having vortex solutions carrying more than one non-Abelian modulus factor have been studied recently [19]. Furthermore, vortices were found to provide us with a deep physical intuition about some well-known correspondence between theories in the four and two dimensions [32, 33, 42]. So far, however, most studies have been limited to the case of $\mathrm{U}(N)$ gauge theories, with a few but notable exceptions [20–22].

In a brief note, some of the present authors have presented a general prescription for constructing the Bogomol'nyi-Prasad-Sommerfield (BPS) vortices in color-flavor locked vacua of a more general class of theories, with a gauge group of the form, $G = G' \times \mathrm{U}(1)$, where G' is any semi-simple group [23]. Some explicit expressions for the moduli matrix construction of the minimum-winding vortex in $\mathrm{SO}(N)$, $\mathrm{USp}(2N)$ models were given there.

It is the purpose of this paper to discuss the properties of the non-Abelian BPS vortices in $\mathrm{SO}(N)$ and $\mathrm{USp}(2N)$ gauge theories in more detail. The moduli space in each case is carefully analyzed, both for the fundamental (or minimal) vortices and for the winding-number two vortices. The study of the non-minimal vortices and their transformation properties is particularly important from the point of view that the latter has a simple group-theoretic nature, in terms of a dual group.

When the model is embedded in a larger, underlying gauge group, spontaneously broken to the model under study, the vortex transformation properties endow the monopoles appearing at the extremes of the vortices with non-Abelian moduli.[1]

The paper is organized as follows. In section 2 the model is presented and the vortex Bogomol'nyi equations are put in a simple form by the introduction of the moduli matrix. The basic characterization of vortices in SO(N) and USp($2N$) theories which follows from this general construction is discussed. Section 3 is dedicated to the study of vortex solutions of the Abrikosov-Nielsen-Olesen (ANO) [1, 2] type (sometimes called local vortices), their moduli space and its structure. We make use of concrete examples (the lowest-rank gauge groups) for the sake of clarity. The analysis is then extended in section 4 to a larger set of BPS-saturated vortex solutions which includes the so-called semi-local vortices [3]. The structure of the moduli space including these points becomes much richer. Again, we discuss in some detail a few concrete cases with the lowest-rank gauge groups. An index theorem for the dimension of the moduli space for vortices with a general gauge group $\mathrm{U}(1) \times G'$ is discussed in appendix A.

Two issues of considerable interest seem to emerge from our study, which are only briefly discussed here. One is the question of the Goddard-Nuyts-Olive-Weinberg(GNOW) quantization/duality of the non-Abelian vortices, which is deeply related to the original problem of understanding non-Abelian *monopoles* [43]. Another is the appearance of "fractional vortices", which seems to be very common when one studies vortices in models other than the U(N) gauge theories. Although the results of the present paper provide us with a concrete starting point and important ingredients for the analysis of these questions, in order to keep the length of the paper to a reasonable size and for the ease of reading, we reserve a more thorough discussion of these two issues for separate, forthcoming papers.

2 Vortex equations and basics

2.1 The moduli matrix and BPS equations

In this section we study vortex solutions in four-dimensional gauge theories with an $\mathrm{SO}(N) \times \mathrm{U}(1)$ or $\mathrm{USp}(N) \times \mathrm{U}(1)$ gauge group,[2] and with N_{F} scalars in the fundamental representation. Sometimes the gauge group will be indicated in a more general way, as $G = G' \times \mathrm{U}(1)$ with G' being any simple group; the prescription for writing down the BPS vortex solutions in all these cases has in fact been given in ref. [23]. However, below we shall concentrate on the gauge groups $\mathrm{SO}(N) \times \mathrm{U}(1)$ and $\mathrm{USp}(N) \times \mathrm{U}(1)$. An integer M will be used to indicate the gauge group, such that $N = 2M$ or $N = 2M + 1$, for even SO(N) and USp(N) or odd SO(N), respectively.

The Lagrangian density reads

$$\mathcal{L} = \mathrm{Tr}_c\left[-\frac{1}{2e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \mathcal{D}_\mu H\,(\mathcal{D}^\mu H)^\dagger - \frac{e^2}{4}\left|X^0 t^0 - 2\xi t^0\right|^2 - \frac{g^2}{4}\left|X^a t^a\right|^2\right], \quad (2.1)$$

[1] The latter has been interpreted as gauge modulations in the dual, confinement phase [16].

[2] The case of local vortices with the gauge groups $\mathrm{SO}(N) \times \mathrm{U}(1)$ has first been considered in ref. [21].

with the field strength, gauge fields and covariant derivative denoted as

$$\begin{aligned} F_{\mu\nu} &= F^0_{\mu\nu} t^0 \,, \qquad F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu \,, \qquad \hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu]\,, \\ A_\mu &= A^a_\mu t^a \,, \qquad \mathcal{D}_\mu = \partial_\mu + i A^0_\mu t^0 + i A^a_\mu t^a \,. \end{aligned} \tag{2.2}$$

A^0_μ is the gauge field associated with U(1) and A^a_μ are the gauge fields of G'. The matter scalar fields are written as an $N \times N_{\rm F}$ complex color (vertical)-flavor (horizontal) mixed matrix H. It can be expanded as

$$X = HH^\dagger = X^0 t^0 + X^a t^a + X^\alpha t^\alpha \,, \quad X^0 = 2\,\mathrm{Tr}_c\left(HH^\dagger t^0\right) \,, \quad X^a = 2\,\mathrm{Tr}_c\left(HH^\dagger t^a\right) \,, \tag{2.3}$$

where the traces with subscript c are over the color indices. e and g are the U(1) and G' coupling constants, respectively, while ξ is a real constant. t^0 and t^a stand for the U(1) and G' generators, respectively, and finally, $t^\alpha \in \mathfrak{g}'_\perp$, where $\mathfrak{g}'_\perp$ is the orthogonal complement of the Lie algebra $\mathfrak{g}'$ in $\mathfrak{su}(N)$. We normalize the generators according to

$$t^0 = \frac{\mathbf{1}_N}{\sqrt{2N}} \,, \qquad \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab} \,. \tag{2.4}$$

We have chosen in eq. (2.1) a particular, critical quartic scalar coupling equal to the (square of the) gauge coupling constants, i.e. the BPS limit. Indeed such a Lagrangian can be regarded as the truncated bosonic sector of an $\mathcal{N} = 2$ supersymmetric gauge theory.[3] The constant ξ would in this case be the Fayet-Iliopoulos parameter. In order to keep the system in the Higgs phase, we take $\xi > 0$. The model has a gauge symmetry acting from the left on H and a flavor symmetry acting from the right. First we note that this theory has a continuous Higgs vacuum which was discussed in detail in ref. [22]. In this paper, we choose to work in a particular point of the vacuum manifold:

$$\langle H \rangle = \frac{v}{\sqrt{N}} \mathbf{1}_N \,, \qquad \xi = \frac{v^2}{\sqrt{2N}} \,, \tag{2.5}$$

namely, in the maximally "color-flavor-locked" Higgs phase of the theory. We have set $N_{\rm F} = N$ which is the minimal number of flavors allowing such a vacuum.[4] The existence of a continuous vacuum degeneracy implies the emergence of vortices of semi-local type as we shall see shortly.

Performing the Bogomol'nyi completion, the energy (tension) reads

$$\begin{aligned} T &= \int d^2x\, \mathrm{Tr}_c \left[\frac{1}{e^2} \left| F_{12} - \frac{e^2}{2}\left(X^0 t^0 - 2\xi t^0\right) \right|^2 + \frac{1}{g^2} \left| \hat{F}_{12} - \frac{g^2}{2} X^a t^a \right|^2 + 4 \left| \bar{\mathcal{D}} H \right|^2 - 2\xi F_{12} t^0 \right] \\ &\geq -\xi \int d^2x\, F^0_{12} \,, \end{aligned} \tag{2.6}$$

[3] The full supersymmetric bosonic sector contains an additional set of squarks in the anti-fundamental representation of the gauge group, and an adjoint scalar field. We can consistently forget about them, as they are trivial on the BPS vortices. Although we shall not make explicit use of any of the consequences of $\mathcal{N} = 2$ supersymmetry (the missing sector is truly relevant at the quantum level), this way of regarding our system is useful for providing a convenient choice of the potential and its stability against radiative corrections.

[4] Notice that this not the minimal choice for the existence of a vacuum which supports BPS vortices. In fact, such a minimal number is $N_{\rm F} = 1$ in the SO case and $N_{\rm F} = 2$ in the USp case. However, in this case there is a residual Coulomb phase. The vortices actually reduce to those appearing in theories with a lower-rank gauge group.

where $\bar{\mathcal{D}} \equiv \frac{\mathcal{D}_1 + i\mathcal{D}_2}{2}$ is used along with the standard complex coordinates $z = x^1 + ix^2$ and all fields are taken to be independent of x^3. When the inequality is saturated (BPS condition), the tension is simply

$$T = 2\sqrt{2N}\pi\xi\nu = 2\pi v^2\nu\,, \qquad \nu = -\frac{1}{2\pi\sqrt{2N}}\int d^2x\, F_{12}^0\,, \tag{2.7}$$

where ν is the U(1) winding number of the vortex. This leads immediately to the BPS equations for the vortex

$$\bar{\mathcal{D}}H = \bar{\partial}H + i\bar{A}H = 0\,, \tag{2.8}$$

$$F_{12}^0 = e^2\left[\mathrm{Tr}_c\left(HH^\dagger t^0\right) - \xi\right]\,, \tag{2.9}$$

$$F_{12}^a = g^2\,\mathrm{Tr}_c\left(HH^\dagger t^a\right)\;. \tag{2.10}$$

The matter BPS equation (2.8) can be solved [9–11] by the Ansatz

$$H = S^{-1}(z,\bar{z})H_0(z)\,, \qquad \bar{A} = -iS^{-1}(z,\bar{z})\bar{\partial}S(z,\bar{z})\,, \tag{2.11}$$

where S belongs to the complexification of the gauge group, $S \in \mathbb{C}^* \times G'^{\mathbb{C}}$. $H_0(z)$, holomorphic in z, is called the *moduli matrix* [12], which contains all moduli parameters of the vortices as will be seen below.

A gauge invariant object can be constructed as $\Omega = SS^\dagger$. It will, however, prove convenient to split this into the U(1) part and the G' part, such that $S = s\,S'$ and analogously $\Omega = \omega\,\Omega'$, $\omega = |s|^2$, $\Omega' = S'S'^\dagger$. In terms of ω the tension (2.7) can be rewritten as

$$T = 2\pi v^2\nu = 2v^2\int d^2x\;\partial\bar{\partial}\log\omega\,, \qquad \nu = \frac{1}{\pi}\int d^2x\,\partial\bar{\partial}\log\omega\,, \tag{2.12}$$

and ν determines the asymptotic behavior of the Abelian field as

$$\omega = ss^\dagger \sim |z|^{2\nu}\,, \qquad \text{for } |z| \to \infty\;. \tag{2.13}$$

The minimal vortex solutions can be written down [23] by making use of the holomorphic invariants for the gauge group G' made of H, which we denote $I^i_{G'}(H)$. If the U(1) charge of the i-th invariant is denoted by n_i, $I^i_{G'}(H)$ satisfies

$$I^i_{G'}(H) = I^i_{G'}\left(s^{-1}S'^{-1}H_0\right) = s^{-n_i}I^i_{G'}(H_0(z))\,, \tag{2.14}$$

while the boundary condition is

$$I^i_{G'}(H)\Big|_{|z|\to\infty} = I^i_{\mathrm{vev}}\,e^{i\nu n_i\theta}\,, \tag{2.15}$$

where $\nu\,n_i$ is the number of *zeros* of $I^i_{G'}$. This leads then to the following asymptotic behavior

$$I^i_{G'}(H_0) = s^{n_i}I^i_{G'}(H) \overset{|z|\to\infty}{\longrightarrow} I^i_{\mathrm{vev}}z^{\nu n_i}\,. \tag{2.16}$$

It implies that $I^i_{G'}(H_0(z))$, being holomorphic in z, are actually polynomials. Therefore $\nu\, n_i$ must be positive integers for all i:

$$\nu\, n_i \in \mathbb{Z}_+ \qquad \to \qquad \nu = \frac{k}{n_0}\,, \qquad k \in \mathbb{Z}_+\,, \tag{2.17}$$

with

$$n_0 \equiv \gcd\left\{n_i\,|I^i_{\rm vev} \neq 0\right\}\,, \tag{2.18}$$

where "gcd" stands for the greatest common divisor. The U(1) gauge transformation $e^{2\pi i/n_0}$ leaves $I^i_{G'}(H)$ invariant and thus the true gauge group is

$$G = \left[\mathrm{U}(1) \times G'\right]/\mathbb{Z}_{n_0}\,, \tag{2.19}$$

where $\mathbb{Z}_{n_0}$ is the center of the group G'. The minimal winding in U(1) found here, $\frac{1}{n_0}$, corresponds to the minimal element of $\pi_1(G) = \mathbb{Z}$, as it represents a minimal loop in the group manifold G. As a result we find the following non-trivial constraints for H_0

$$I^i_{G'}(H_0) = I^i_{\rm vev}\, z^{\frac{kn_i}{n_0}} + \mathcal{O}\left(z^{\frac{kn_i}{n_0}-1}\right)\,. \tag{2.20}$$

Let us now obtain the explicit constraints for the gauge groups SO and USp. The invariants are

$$\left(I_{SO,USp}\right)^r{}_s = (H^{\rm T}JH)^r{}_s\,, \qquad 1 \le r \le s \le N\,, \tag{2.21}$$

which finally yields what we call the weak constraint for the moduli matrix,

$$H_0^{\rm T}(z)JH_0(z) = z^{\frac{2k}{n_0}}J + \mathcal{O}\left(z^{\frac{2k}{n_0}-1}\right)\,. \tag{2.22}$$

Here J is the invariant tensor of G':

$$J_{\rm even} = \begin{pmatrix} 0 & \mathbf{1}_M \\ \epsilon\,\mathbf{1}_M & 0 \end{pmatrix}\,, \qquad J_{\rm odd} = \begin{pmatrix} 0 & \mathbf{1}_M & 0 \\ \mathbf{1}_M & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{2.23}$$

where in the first matrix $\epsilon = +1$ for SO($2M$) and $\epsilon = -1$ for USp($2M$),[5] while the second matrix is for the SO($2M+1$) theory. The integer n_0 for each group is listed in table 1. Vortices represented by eq. (2.22) include also semi-local vortices.

In terms of Ω the BPS-equations (for the gauge-fields) (2.9) and (2.10) can be expressed as

$$\partial\bar{\partial}\log\omega = \frac{m_e^2}{4}\left(1 - \frac{1}{N\omega}\mathrm{Tr}_c\left(\Omega_0\Omega'^{-1}\right)\right)\,, \tag{2.24}$$

$$\bar{\partial}\left(\Omega'\partial\Omega'^{-1}\right) = \frac{m_g^2}{8\,\omega}\left(\Omega_0\Omega'^{-1} - J^\dagger(\Omega_0\Omega'^{-1})^{\rm T}J\right)\,, \tag{2.25}$$

[5]The symbol ϵ will appear many times below. It will always take one of the two values, depending on the choice of the gauge group

	SO$(2M)$	USp$(2M)$	SO$(2M+1)$
n_0	2	2	1

Table 1. n_0 for SO(N) and USp$(2M)$

where $\Omega_0 \equiv \frac{N}{v^2} H_0 H_0^\dagger$ and $m_e = \frac{ev}{\sqrt{N}}$, $m_g = \frac{gv}{\sqrt{N}}$ are masses around the vacuum (2.5). The equations (2.24) and (2.25) are called *master equations* for the gauge group $G' =$ SO(N) and USp$(2M)$ with the respective invariant tensor J. Both sides of these equations transform covariantly under the following transformation:

$$S(z,\bar{z}) \to V_e(z) V'(z) S(z,\bar{z})\,, \quad H_0(z) \to V_e(z) V'(z) H_0(z)\,, \quad V_e(z) \in \mathbb{C}^*\,,\ V'(z) \in G'^{\mathbb{C}}. \tag{2.26}$$

This transformation does not change the original fields H and A (see equation (2.11)). Therefore, the solutions to the equations (2.24) and (2.25) are equivalent if they are related by the transformation (2.26). We denote this the V*-equivalence relation*. The master equations (2.24) and (2.25) should be solved such that the solution approaches the vacuum configuration at the boundary $|z| \to \infty$. Therefore, one must enforce the following asymptotic behavior on[6] $\Omega = \omega \Omega'$,

$$\log \Omega = \log \Omega_\infty + \mathcal{O}\left(\frac{1}{m_{e,g} z}, \frac{1}{m_{e,g} \bar{z}} \right). \tag{2.27}$$

Here the leading contribution $\Omega_\infty = \omega_\infty \Omega'_\infty$ is given as the unique solution to the D-term conditions $X^0 = X^a = 0$ with a given $H_0(z)$. They are obtained by the Kähler quotient method and are found for the gauge groups $G' = \mathrm{SO}(N), \mathrm{USp}(N)$ in ref. [22] to be:

$$\Omega'_\infty = H_0(z) \frac{\mathbf{1}_N}{\sqrt{I_{G'}^\dagger I_{G'}}} H_0(z)^\dagger, \quad \omega_\infty = \frac{1}{v^2} \mathrm{Tr}\left[\sqrt{I_{G'}^\dagger I_{G'}} \right], \tag{2.28}$$

where the G'-invariant $I_{G'} = I_{G'}(H_0) = H_0^{\mathrm{T}}(z) J H_0(z)$. With this boundary condition, the master equations are expected to have a unique (and smooth) solution with a given $H_0(z)$. Namely, we expect that vortex configurations are completely characterized by $H_0(z)$. The validity of this expectation will be discussed in section 4.1.

2.2 GNOW quantization for non-Abelian vortices

Our task is to find all possible moduli matrices which satisfy the weak condition (2.22). In general this is not easy. But certain special solutions can be found readily, and each such solution is characterized by a *weight vector of the dual group*, and are labelled by a set of integers ν_a $(a = 1, \ldots, \mathrm{rank}(G'))$

$$H_0(z) = z^{\nu \mathbf{1}_N + \nu_a \mathcal{H}_a} \in \mathrm{U}(1)^{\mathbb{C}} \times G'^{\mathbb{C}}, \tag{2.29}$$

[6] For vortices satisfying the strong condition (2.64), Ω_∞ reduces to Ω_0 and the next to leading terms of $\log \Omega$ are $\mathcal{O}\left(e^{-m_{e,g}|z|}\right)$ as will be explained later.

where $\nu = k/n_0$ is the U(1) winding number and $\mathcal{H}_a$ are the generators of the Cartan subalgebra of $\mathfrak{g}'$. These special solutions satisfy the strong condition (2.64), given below, with $z_i = 0$. H_0 must be holomorphic in z and *single-valued*, which gives the constraints for a set of integers ν_a

$$(\nu \mathbf{1}_N + \nu_a \mathcal{H}_a)_{ll} \in \mathbb{Z}_{\geq 0} \quad \forall\, l \, . \tag{2.30}$$

Suppose that we now consider scalar fields in an r-representation of G'. The constraint is equivalent to

$$\nu + \nu_a \mu_a^{(i)} \in \mathbb{Z}_{\geq 0} \quad \forall\, i \, , \tag{2.31}$$

where $\vec{\mu}^{(i)} = \mu_a^{(i)}$ $(i = 1, 2, \ldots, \dim(r))$ are the weight vectors for the r-representation of G'. Subtracting pairs of adjacent weight vectors, one arrives at the quantization condition

$$\vec{\nu} \cdot \vec{\alpha} \in \mathbb{Z} \, , \tag{2.32}$$

for every root vector α of G'.

Eq. (2.32) is formally identical to the well-known Goddard-Nuyts-Olive-Weinberg (GNOW) quantization condition [43] for the monopoles, and to the vortex flux quantization rule found in ref. [44]. There is however a crucial difference here, as compared to the case of [43] or [44]. Because of an exact flavor (color-flavor diagonal $G_{\mathrm{C+F}}$) symmetry present here, which is broken by individual vortex solutions, our vortices possess continuous moduli. As will be seen later, at least in the local case these moduli are normalizable, and there are no conceptual problems in their quantization. On the contrary, vortices in ref. [44] do not have any continuous modulus, while in the case of "non-Abelian monopoles" [43] these interpolating modes suffer from the well-known problems of non-normalizability. Another way the latter difficulty manifests itself is that the naïve "unbroken" group cannot be defined globally due to a topological obstruction [45] in the monopole backgrounds.

The solution of the quantization condition (2.32) is that

$$\tilde{\vec{\mu}} \equiv \vec{\nu}/2 \, , \tag{2.33}$$

is any of the *weight vectors* of the dual group of G'. The dual group, denoted as $\tilde{G}'$, is defined by the dual root vectors [43]

$$\vec{\alpha}^* = \frac{\vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} \, . \tag{2.34}$$

We show examples of dual pairs of groups G', $\tilde{G}'$ in table 2. Note that (2.31) is stronger than (2.32), it has to be zero or a positive integer. This positive quantization condition allows for only a few weight vectors. For concreteness, let us consider scalar fields in the fundamental representation, and choose a basis where the Cartan generators of $G' = \mathrm{SO}(2M), \mathrm{SO}(2M+1), \mathrm{USp}(2M)$ are given by

$$\mathcal{H}_a = \mathrm{diag}\left(\underbrace{0, \ldots, 0}_{a-1}, \frac{1}{2}, \underbrace{0, \ldots, 0}_{M-1}, -\frac{1}{2}, 0, \ldots, 0 \right) , \tag{2.35}$$

G'	$\tilde{G}'$
SU(N)	SU(N)/$\mathbb{Z}_N$
U(N)	U(N)
SO($2M$)	SO($2M$)
USp($2M$)	SO($2M+1$)
SO($2M+1$)	USp($2M$)

Table 2. Some pairs of dual groups

with $a = 1, \ldots, M$. In this basis, special solutions H_0 have the form[7] for $G' = \mathrm{SO}(2M)$ and $\mathrm{USp}(2M)$

$$H_0^{(\tilde{\mu}_1,\ldots,\tilde{\mu}_M)} = \mathrm{diag}\left(z^{k_1^+}, \ldots, z^{k_M^+}, z^{k_1^-}, \ldots, z^{k_M^-}\right) , \tag{2.36}$$

while for $\mathrm{SO}(2M+1)$

$$H_0^{(\tilde{\mu}_1,\ldots,\tilde{\mu}_M)} = \mathrm{diag}\left(z^{k_1^+}, \ldots, z^{k_M^+}, z^{k_1^-}, \ldots, z^{k_M^-}, z^k\right) , \tag{2.37}$$

where $k_a^\pm = \nu \pm \tilde{\mu}_a$.

For example, in the cases of $G' = \mathrm{SO}(4), \mathrm{USp}(4)$ with a $\nu = 1/2$ vortex, there are four special solutions with $\vec{\tilde{\mu}} = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$

$$H_0^{(\frac{1}{2},\frac{1}{2})} = \mathrm{diag}(z, z, 1, 1) = z^{\frac{1}{2}\mathbf{1}_4 + 1\cdot\mathcal{H}_1 + 1\cdot\mathcal{H}_2}, \tag{2.38}$$
$$H_0^{(\frac{1}{2},-\frac{1}{2})} = \mathrm{diag}(z, 1, 1, z) = z^{\frac{1}{2}\mathbf{1}_4 + 1\cdot\mathcal{H}_1 - 1\cdot\mathcal{H}_2}, \tag{2.39}$$
$$H_0^{(-\frac{1}{2},\frac{1}{2})} = \mathrm{diag}(1, z, z, 1) = z^{\frac{1}{2}\mathbf{1}_4 - 1\cdot\mathcal{H}_1 + 1\cdot\mathcal{H}_2}, \tag{2.40}$$
$$H_0^{(-\frac{1}{2},-\frac{1}{2})} = \mathrm{diag}(1, 1, z, z) = z^{\frac{1}{2}\mathbf{1}_4 - 1\cdot\mathcal{H}_1 - 1\cdot\mathcal{H}_2}. \tag{2.41}$$

These four vectors are the same as the weight vectors of two Weyl spinor representations $\mathbf{2} \oplus \mathbf{2}'$ of $\tilde{G}' = \mathrm{SO}(4)$ for $G' = \mathrm{SO}(4)$, and the same as those of the Dirac spinor representation $\mathbf{4}$ of $\tilde{G}' = \mathrm{Spin}(5)$ for $G' = \mathrm{USp}(4)$.

The second example is $G' = \mathrm{SO}(5)$ with $\nu = 1$. We have nine special points which are described by $\vec{\tilde{\mu}} = (0,0)$ and $(1,0), (0,1), (-1,0), (0,-1)$ and $(1,1), (1,-1), (-1,1), (-1,-1)$ and thus correspond to

$$H_0^{(0,0)} = \mathrm{diag}(z, z, z, z, z) = z^{1\cdot\mathbf{1}_5 + 0\cdot\mathcal{H}_1 + 0\cdot\mathcal{H}_2}, \tag{2.42}$$
$$H_0^{(1,0)} = \mathrm{diag}(z^2, z, 1, z, z) = z^{1\cdot\mathbf{1}_5 + 2\cdot\mathcal{H}_1 + 0\cdot\mathcal{H}_2}, \tag{2.43}$$
$$H_0^{(0,1)} = \mathrm{diag}(z, z^2, z, 1, z) = z^{1\cdot\mathbf{1}_5 + 0\cdot\mathcal{H}_1 + 2\cdot\mathcal{H}_2}, \tag{2.44}$$
$$H_0^{(-1,0)} = \mathrm{diag}(1, z, z^2, z, z) = z^{1\cdot\mathbf{1}_5 - 2\cdot\mathcal{H}_1 + 0\cdot\mathcal{H}_2}, \tag{2.45}$$
$$H_0^{(0,-1)} = \mathrm{diag}(z, 1, z, z^2, z) = z^{1\cdot\mathbf{1}_5 + 0\cdot\mathcal{H}_1 - 2\cdot\mathcal{H}_2}, \tag{2.46}$$
$$H_0^{(1,1)} = \mathrm{diag}(z^2, z^2, 1, 1, z) = z^{1\cdot\mathbf{1}_5 + 2\cdot\mathcal{H}_1 + 2\cdot\mathcal{H}_2}, \tag{2.47}$$

[7] The integers $k_a^\pm$ and k here coincide with $n_a^\pm$ and $n^{(0)}$, respectively, of ref. [21].

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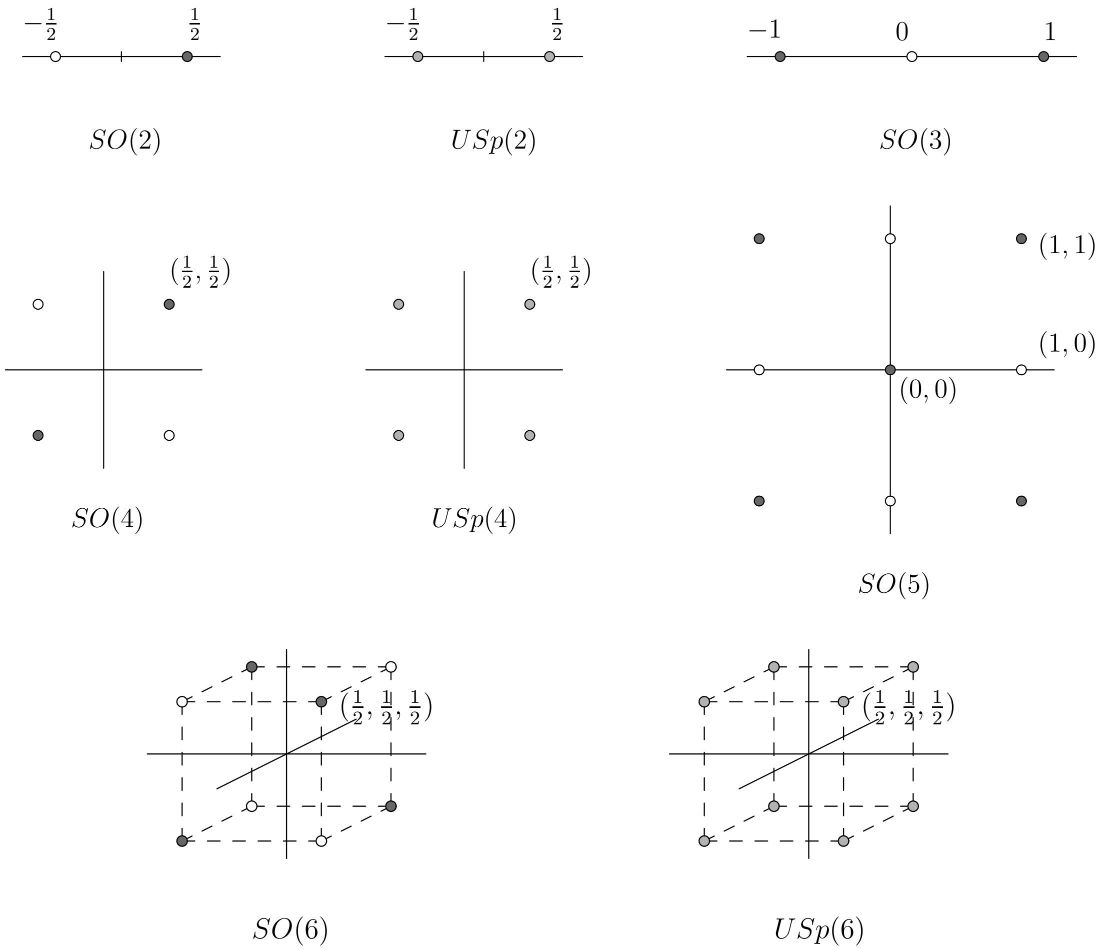

Figure 1. The special points for the $k = 1$ vortex.

$$H_0^{(1,-1)} = \mathrm{diag}(z^2, 1, 1, z^2, z) = z^{1 \cdot \mathbf{1}_5 + 2 \cdot \mathcal{H}_1 - 2 \cdot \mathcal{H}_2}, \tag{2.48}$$
$$H_0^{(-1,1)} = \mathrm{diag}(1, z^2, z^2, 1, z) = z^{1 \cdot \mathbf{1}_5 - 2 \cdot \mathcal{H}_1 + 2 \cdot \mathcal{H}_2}, \tag{2.49}$$
$$H_0^{(-1,-1)} = \mathrm{diag}(1, 1, z^2, z^2, z) = z^{1 \cdot \mathbf{1}_5 - 2 \cdot \mathcal{H}_1 - 2 \cdot \mathcal{H}_2}. \tag{2.50}$$

These nine vectors are the same as the weight vectors of the vector representation **4** and the antisymmetric representation **5** of the dual group $\tilde{G}' = \mathrm{USp}(4)$. The weight vectors corresponding to the $k = 1$ vortex in various gauge groups are given in figure 1.

2.3 $\mathbb{Z}_2$ parity

As discussed in ref. [21], the vortices in $G' = \mathrm{SO}(N)$ theory are characterized by the first homotopy group

$$\pi_1\left(\frac{\mathrm{SO}(N) \times \mathrm{U}(1)}{\mathbb{Z}_{n_0}}\right) = \mathbb{Z} \times \mathbb{Z}_2\,, \qquad n_0 = 1 \;\; (N \text{ odd})\,, \quad n_0 = 2 \;\; (N \text{ even})\,, \tag{2.51}$$

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$Q_{\mathbb{Z}_2}$
$\frac{1}{2}$	$\frac{1}{2}$	$+1$
$\frac{1}{2}$	$-\frac{1}{2}$	-1
$-\frac{1}{2}$	$\frac{1}{2}$	-1
$-\frac{1}{2}$	$-\frac{1}{2}$	$+1$

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$Q_{\mathbb{Z}_2}$
0	0	$+1$
$\pm\{1$	$0\}$	-1
$\pm\{1$	$\pm1\}$	$+1$

Table 3. $k = 1$ SO(4) vortices (left), $k = 1$ SO(5) or $k = 2$ SO(4) (right).

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$\tilde{\mu}_3$	$Q_{\mathbb{Z}_2}$
$\pm\{\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\}$	±1
$\pm\{\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}\}$	∓1

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$\tilde{\mu}_3$	$Q_{\mathbb{Z}_2}$
0	0	0	-1
$\pm\{1$	0	$0\}$	$+1$
$\pm\{1$	1	$0\}$	-1
$\pm\{1$	-1	$0\}$	-1
$\pm\{1$	1	$1\}$	$+1$
$\pm\{-1$	1	$1\}$	$+1$

Table 4. $k = 1$ SO(6) cases (left), $k = 1$ SO(7) or $k = 2$ SO(6) (right).

while those of $G' = \mathrm{USp}(2M)$ theory correspond to non-trivial elements of

$$\pi_1\left(\frac{\mathrm{USp}(2M)\times \mathrm{U}(1)}{\mathbb{Z}_2}\right) = \mathbb{Z}\ . \tag{2.52}$$

The vortices in $G' = \mathrm{SO}(N)$ carry a $\mathbb{Z}_2$ charge in addition to the usual additive vortex charges. The $\mathbb{Z}_2$ charge can be seen from the dual weight vector $\vec{\tilde{\mu}}$. As a simple example, let us consider the case of SO(4). The dual weight vectors are listed in table 3. Let us compare two states: namely $(\tilde{\mu}_1, \tilde{\mu}_2) = (1/2, 1/2)$ and $(\tilde{\mu}_1, \tilde{\mu}_2) = (1/2, -1/2)$. The difference between them is $\delta(\tilde{\mu}_1, \tilde{\mu}_2) = (0, 1)$: thus one of them can be obtained from the other by a 2π rotation in the (24)-plane in SO(4). As a path from unity to a 2π rotation is a non-contractible loop, they have different $\mathbb{Z}_2$ charges.

On the other hand, the difference between $(\tilde{\mu}_1, \tilde{\mu}_2) = (1/2, 1/2)$ and $(\tilde{\mu}_1, \tilde{\mu}_2) = (-1/2, -1/2)$ is $\delta(\tilde{\mu}_1, \tilde{\mu}_2) = (1, 1)$, hence this is homotopic to the trivial element of $\mathbb{Z}_2$. Therefore, the vortices can be classified by the $\mathbb{Z}_2$-parity, $Q_{\mathbb{Z}_2} = \pm1$. In figures 1 and 4, the dark points correspond to vortices with $Q_{\mathbb{Z}_2} = +1$ while the empty circles correspond to those with $Q_{\mathbb{Z}_2} = -1$.

The $\mathbb{Z}_2$ parity of each special point is defined, in general, as follows:

$$Q_{\mathbb{Z}_2}(k_i^+, k_i^-) = (+)^{\sum_i k_i^+} \times (-)^{\sum_i k_i^-} = (-)^{\sum_i k_i^-}\,, \tag{2.53}$$

or equivalently in terms of the weight vectors:

$$Q_{\mathbb{Z}_2}(H_0^{(\tilde{\mu}_i,\ldots,\tilde{\mu}_M)}) = (-)^{\nu M - \sum_i \tilde{\mu}_i}. \tag{2.54}$$

2.4 Local versus semi-local vortices

One is often interested in knowing which of the moduli parameters describe the so-called local (or the ANO-) vortices [1, 2]), which have the profile functions with exponential tails.

For example, the thoroughly studied $\mathrm{U}(N)$ non-Abelian vortices are of the local type when the model has a unique vacuum: this is indeed the case when the number of flavors is the minimal one, i.e. just sufficient for the color-flavor locked vacuum ($N_\mathrm{F} = N$ Higgs fields in the $\mathbf{N}$ representation of $\mathrm{SU}(N)$). For N_F greater than N, the vacuum moduli space contains continuous moduli $Gr_{N_\mathrm{F},N} \simeq \mathrm{SU}(N_\mathrm{F})/[\mathrm{SU}(N_\mathrm{F}-N)\times\mathrm{SU}(N)\times\mathrm{U}(1)]$ and, as a consequence, the generic non-Abelian vortex solution is of the "semi-local" type [3, 46], with power-like tails.[8] A characteristic feature of the semi-local vortices is their size moduli, which are non-normalizable [17, 18]. A lesson from the $\mathrm{U}(N)$ non-Abelian vortices is that the semi-local vortices become local (ANO-like) vortices, when all the size moduli are set to zero.

Our model with $G' = \mathrm{SO}(N)$ or $\mathrm{USp}(2M)$, even with our choice $N_\mathrm{F} = N$, that is the minimum number of flavors that allows for a color-flavor locked vacua, possesses always a non-trivial vacuum moduli space. In fact, in the class of theories considered here, its dimension is given by the following general formula

$$\dim_\mathbb{C}\left[\mathcal{M}_\mathrm{vac}\right] = N N_\mathrm{F} - \dim_\mathbb{C}\left[\mathrm{U}(1)^\mathbb{C}\times G'^\mathbb{C}\right] > 0\ . \tag{2.55}$$

This strongly suggests that even for $N_\mathrm{F} = N$, generic configurations are of the "semi-local" type. The Kähler metric and its potential on the vacuum moduli space have been obtained in ref. [22].

The distinction between local and semi-local vortices can be made by using the moduli matrix. In order to see this, the asymptotic behavior of the configurations must be clarified. First note that the vacuum moduli spaces of our models are Kähler manifolds $\mathcal{M}_\mathrm{vac}$ and our gauge theories reduce to non-linear sigma models whose target space is $\mathcal{M}_\mathrm{vac}$, when the gauge couplings are sent to infinity. In this limit, vortices generally reduce to the so-called sigma model lumps [47] (sometimes also called two-dimensional Skyrmions or sigma model instantons) characterized by

$$\pi_2(\mathcal{M}_\mathrm{vac})\ ,$$

i.e. a wrapping around a 2-cycle inside $\mathcal{M}_\mathrm{vac}$. By rescaling sizes, taking the strong coupling limit can be interpreted as picking up the asymptotic behavior, and thus, even for a finite gauge coupling, asymptotic configurations of semi-local vortices are well-approximated by lumps [46].

Consider the lump solutions of the non-linear sigma model on $\mathcal{M}_\mathrm{vac}$. Let us take holomorphic G-invariants $\{I_G^I\}$ as inhomogeneous coordinates of $\mathcal{M}_\mathrm{vac}$ and denote its Kähler potential by $K = K(I_G, I_G^*)$. A lump solution is then given by a holomorphic map

$$z\in\mathbb{C} \quad\rightarrow\quad I_G^I = f^I(z) \in \mathcal{M}_\mathrm{vac}\ . \tag{2.56}$$

with single-valued functions $\{f^I(z)\}$. For finite-energy solutions, the boundary $|z| = \infty$ is mapped to a single point $I_G^I = v^I \in \mathcal{M}_\mathrm{vac}$. So the maps $\{f^I(z)\}$ are asymptotically of the form

$$f^I(z) = v^I + \frac{u^I}{z} + \mathcal{O}\left(z^{-2}\right),\quad u^I\in\mathbb{C}\ . \tag{2.57}$$

[8] "Local vortex" and "semi-local vortex" are clearly misnomers, but as they seem to have stuck among the experts in the field, we shall use them in this paper.

The corresponding energy density $\mathcal{E}$ has a power behavior

$$\mathcal{E} = 2K_{J\bar{J}}(I_G, \bar{I}_G)\,\partial I_G^J(z)\,\bar{\partial}\bar{I}_G^{\bar{J}}(\bar{z}) = \frac{2}{|z|^4}\,K_{J\bar{J}}(v,\bar{v})\,u^J\bar{u}^{\bar{J}} + \mathcal{O}\left(|z|^{-5}\right)\,, \tag{2.58}$$

where we assume that $\{I_G^I\}$ is a local coordinate system in the vicinity of the point $I_G^I = v^I$ and the manifold is smooth at that point. As mentioned above, this asymptotic behavior is valid for that of the vortices as well. Since $\{I_G^I\} \simeq \{I_{G'}^i\}/\mathrm{U}(1)^{\mathbb{C}}$ in the case $G = G' \times \mathrm{U}(1)$, the holomorphic maps and the moduli matrix are related by

$$\{f^I(z)\} \simeq \{I_{G'}(z)\}/\sim\,, \tag{2.59}$$

where "$\sim$" is defined as the equivalence relation

$$I_{G'}^i(z) \sim P(z) I_{G'}^i(z)\,, \qquad \text{with } P(z) \in \mathbb{C}[z]\ . \tag{2.60}$$

Hence, the asymptotic tail of the configurations is generically power-like, i.e. the generic vortices are of the semi-local type.

Although this is in general the case, it might happen that all the holomorphic functions $\{I_{G'}^i(H_0(z))\}$ have common zeros and that the quotient above is ill-defined. In such a case, from the point of view of $f^I(z)$, we completely lose the information about the common zeros accompanied by some vorticity. Namely, the signature of the corresponding vortices vanishes from their polynomial tails and $\pi_2(\mathcal{M}_{\mathrm{vac}})$ becomes trivial.[9] Specifically, it can happen that all the holomorphic invariants are proportional to a polynomial $P(z)$:

$$f^I(z) = \text{const.} \quad \Leftarrow \quad I_{G'}^i(H_0(z)) = P(z)^{\frac{n_i}{n_0}} \quad \text{for all } i\,, \tag{2.61}$$

or possibly that there exists only one such holomorphic invariant. In the case of the $\mathrm{U}(N)$, with $N_{\mathrm{F}} = N$ i.e. the model considered earlier, $\mathcal{M}_{\mathrm{vac}}$ is just a single point. Even in the SO and USp cases, we do not consider any non-trivial element of the second homotopy group of $\mathcal{M}_{\mathrm{vac}}$ but we fix a point of $\mathcal{M}_{\mathrm{vac}}$ at $|z| \to \infty$. Therefore, one must return to the master equations to examine the asymptotic behavior. The moduli matrix satisfying eq. (2.61) could be transformed to a trivial one such that $\Omega_0 = \mathbf{1}_N$ in eqs. (2.24) and (2.25), by using an extended V-transformation allowing for negative powers of z, with a singular determinant $\det\left(V(z)\right) = P(z)^{-1}$. After this operation the master equation would take the form of a Liouville-type equation with point-like sources;[10] hence the asymptotic tail is indeed exponential. In other words, the conditions (2.61) mean that the (static) vortex is decoupled from any massless mode in the Higgs vacuum and hence the dominant contribution to its configuration comes from massive modes in the bulk. The corresponding vortices are purely of local type. Conversely, we can clearly identify a local vortex and its position by

[9] The price of the loss of vorticity in the map (2.56) is the appearance of small lump singularities, which manifest themselves as spikes (delta functions) in the energy density.

[10] In the well-known Abelian case $G = \mathrm{U}(1)$, this transformed master equation is nothing but Taubes' equation. This transformation for non-Abelian cases means that all information about orientational moduli are also localized at the zeros, in other words, the moduli matrix can be reconstructed from the data at the zeros in the case of local vortices [24]. For semi-local vortices, this is clearly not the case.

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looking at common zeros, although a composite state of a semi-local vortex and a local vortex also has a polynomial tail. The above observations can briefly be summarized as follows. The asymptotic behavior of a vortex is classified by the lightest modes in the bulk coupled to its configuration. To summarize, a vortex is necessarily of the local type, when the vacuum moduli space is just a point (i.e. a unique vacuum). Semi-local vortices are present only if the vacuum moduli space is non-trivial (i.e. having continuous moduli).

Once we have clarified the origin of the of polynomial tails, it is easier to identify the non-normalizable modes and the results in ref. [22] for lumps can be readily applied to vortices. *Semi-local vortices always have non-normalizable moduli, which live on the tangent bundle of the moduli space of vacua*[11]

$$(v^I, u^I) \in T\mathcal{M}_{\rm vac} \ . \tag{2.62}$$

In our case, $G' = \mathrm{SO}(N), \mathrm{USp}(N)$, with the common U(1) charge of the scalar fields H, all the $G^{\mathbb{C}}$ invariants $I^I_G(H)$ can be written using the meson $I_{SO,USp}$ in eq. (2.21). For instance, since $\mathrm{Tr}[I_{SO,USp}] \neq 0$ in the chosen vacuum, we can construct

$$I_G^{(r,s)}(H) \equiv \frac{(I_{SO,USp}(H))^r{}_s}{\mathrm{Tr}[I_{SO,USp}(H)]} = \frac{\left(H^{\mathrm{T}}JH\right)^r{}_s}{\mathrm{Tr}[H^{\mathrm{T}}JH]}\,, \qquad 1 \le r \le s \le N \ . \tag{2.63}$$

The condition for (winding k) local vortices is thus:

$$I_{SO,USp}(H_0) = H_0^{\mathrm{T}}(z)JH_0(z) = \left(\prod_{i=1}^{k}(z-z_i)^{\frac{2}{n_0}}\right) J \ . \tag{2.64}$$

This will be called the strong condition, in contrast to the weak condition (2.22) which characterizes a more general class of solutions including semi-local vortices.

In the next section we will discuss moduli spaces defined by requiring the strong condition. One can regard this condition being physically required by modifying our model in such a way that the continuous directions of the vacuum are indeed being lifted. For instance, it is not difficult to add an appropriate superpotential δW to our model, introducing a chiral multiplet A which is a traceless N-by-N matrix taking value in the $\mathfrak{usp}$ ($\mathfrak{so}$) algebra in the SO case (USp case), viz. $A^{\mathrm{T}}J = JA$, and having a U(1) charge -2:

$$\delta W \propto \mathrm{Tr}[AH^{\mathrm{T}}JHJ]\,, \tag{2.65}$$

however such a term would nevertheless reduce the amount of supersymmetry. As we will see in some cases, the strong condition can give rise to singularities in the moduli space, which will be inherited into the target space of an effective action for the local vortices.

3 Local vortices and their orientational moduli

In this section we study local non-Abelian vortices in detail leaving the analyses of semi-local vortices for the next section. The *local* non-Abelian vortices carry non-Abelian charges

[11] v^I are nothing but vacuum moduli and all of the u^I's are not always independent and consist of overall semi-local moduli like an overall size modulus. The interpretation as a tangent bundle can be derived from eq. (2.57)

under the color-flavor symmetry group. The corresponding moduli parameters are referred to as the internal orientations (or orientational modes) of the vortices. Let us consider a single local vortex. The strong condition is

$$H_0^{\rm T}(z) J H_0(z) = (z-z_0)^{\frac{2}{n_0}} J \ . \tag{3.1}$$

The parameter z_0 represents the vortex center and is a part of the vortex moduli. Fixing $z_0 = 0$, the solutions to the above condition still possess the orientational modes. In fact, once a moduli matrix satisfying eq. (3.1) has been found, other solutions are readily obtained by acting on it with the color-flavor symmetry transformations $G'_{\rm C+F}$:

$$H_0'(z) \equiv H_0(z) U \ , \quad U \in G'_{\rm C+F} \ . \tag{3.2}$$

However, $H_0(z)$ is defined only *modulo* V-equivalence, therefore if there exists a V-transformation such that

$$V(z) H_0'(z) = H_0(z) \ , \quad V(z) \in G'^{\mathbb{C}} \ , \tag{3.3}$$

then $H_0'(z)$ and $H_0(z)$ should be regarded as physically the same configuration. Therefore, in order to identify the orientational moduli, one needs to identify the flavor rotations which cannot be undone by any V-transformation. In the case of $k=1$ local vortices with $G' = {\rm SO}(2M), {\rm USp}(2M)$, this discussion is sufficient to describe the moduli spaces completely. In the ${\rm SO}(2M+1)$ case, and for higher-winding vortices, however, this is not the case. It is there that the moduli matrix formalism shows its power.

3.1 The single ($k=1$) local vortex for $G' = {\rm SO}(2M), {\rm USp}(2M)$

The strong condition (3.1) with $n_0 = 2$ is satisfied by all special moduli matrices given in eq. (2.36). For simplicity, let us start with the moduli matrix described by the dual weight vector $\vec{\mu} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, i.e.

$$H_0^{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})}(z) = {\rm diag}(\,\underbrace{z,\ldots,z}_{M},\ \underbrace{1,\ldots,1}_{M}\,) \ . \tag{3.4}$$

The color-flavor rotation $G'_{\rm C+F}$ generates other moduli matrices in a $G'_{\rm C+F}/{\rm U}(M)$-orbit. It is obvious that the action of the ${\rm U}(M)$ subgroup of $G' = {\rm SO}(2M), {\rm USp}(2M)$

$$U_0 = \begin{pmatrix} u^{\rm T} & \\ & u^{-1} \end{pmatrix} \in G'_{\rm C+F} \ , \quad u \in {\rm U}(M) \ , \tag{3.5}$$

can be undone by a V-transformation (2.26) due to the fact that $H_0^{(\frac{1}{2},\ldots,\frac{1}{2})} U_0 = U_0 H_0^{(\frac{1}{2},\ldots,\frac{1}{2})} \simeq H_0^{(\frac{1}{2},\ldots,\frac{1}{2})}$. Therefore, we find the orientational moduli as parametrizing the following spaces [23]

$$\mathcal{M}_{\rm ori} = \frac{G'_{\rm C+F}}{{\rm U}(M)_{\rm C+F}} = \frac{{\rm SO}(2M)}{{\rm U}(M)} \quad \text{or} \quad \frac{{\rm USp}(2M)}{{\rm U}(M)} \ , \tag{3.6}$$

both of which are Hermitian symmetric spaces [48, 49]. The real dimension of the moduli spaces is $M(2M \mp 1) - M^2 + 2 = M(M \mp 1) + 2$. Where the additional dimension two corresponds to the position of the vortex.

In order to see explicitly $G'_{\rm C+F}/{\rm U}(M)$, let us take the following element of G'

$$U = \begin{pmatrix} \mathbf{1}_M & -b^\dagger_{A,S} \\ & \mathbf{1}_M \end{pmatrix} \begin{pmatrix} \sqrt{\mathbf{1}_M + b^\dagger_{A,S} b_{A,S}} & \\ & \left(\sqrt{\mathbf{1}_M + b_{A,S}\, b^\dagger_{A,S}}\right)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_M & \\ b_{A,S} & \mathbf{1}_M \end{pmatrix}, \tag{3.7}$$

where b_S (b_A) is an arbitrary M-by-M symmetric (antisymmetric)[12] matrix for the SO$(2M)$ (USp$(2M)$) case. The first two matrices in U can be eliminated by V-transformations, such that the action of U brings the moduli matrix $H_0^{(\frac{1}{2},\dots,\frac{1}{2})}$ to the following form

$$H_0^{(\frac{1}{2},\dots,\frac{1}{2})}(z)U \xrightarrow{V} H_0^{(\frac{1}{2},\dots,\frac{1}{2})}(z;b_{A,S}) \equiv \begin{pmatrix} z\mathbf{1}_M & \\ b_{A,S} & \mathbf{1}_M \end{pmatrix} = \begin{pmatrix} z\mathbf{1}_M & \\ & \mathbf{1}_M \end{pmatrix} \begin{pmatrix} \mathbf{1}_M & \\ b_{A,S} & \mathbf{1}_M \end{pmatrix}. \tag{3.8}$$

We denote the patch described by the above moduli matrix the $(\frac{1}{2},\dots,\frac{1}{2})$-patch of the manifold $G'_{\rm C+F}/{\rm U}(M)$. The complex parameters in the $M \times M$ matrix $b_{A,S}$ are the (local) inhomogeneous coordinates of $\mathcal{M}_{\rm ori}$. Indeed, the moduli matrix has $\frac{M(M\mp 1)}{2} + 1$ complex parameters which is in fact the dimension of the moduli space as will be demonstrated in section 4.1. This in turn implies that, in the present case, the moduli space for the local vortex is entirely generated by a G' orbit, except for the position moduli.

By a similar argument we find 2^M patches, starting from the special points $\vec{\mu} = (\pm\frac{1}{2},\dots,\pm\frac{1}{2})$ given in eq. (2.36). Indeed, this can easily be done by means of permutations, e.g.

$$H_0^{(\overbrace{-\frac{1}{2},\dots,-\frac{1}{2}}^{r},\overbrace{\frac{1}{2},\dots,\frac{1}{2}}^{M-r})}(z;b_{A,S}) = P_r^{-1} H_0^{(\frac{1}{2},\dots,\frac{1}{2})}(z;b_{A,S}) P_r\,, \tag{3.9}$$

where the permutation matrix is

$$P_r \equiv \begin{pmatrix} \mathbf{0}_r & & \epsilon\,\mathbf{1}_r & \\ & \mathbf{1}_{M-r} & & \mathbf{0}_{M-r} \\ \mathbf{1}_r & & \mathbf{0}_r & \\ & \mathbf{0}_{M-r} & & \mathbf{1}_{M-r} \end{pmatrix}, \qquad P_r^{\rm T} J P_r = J\,. \tag{3.10}$$

One can easily check that the constraint

$$\left[P_r^{-1} H_0^{(\frac{1}{2},\dots,\frac{1}{2})} P_r\right]^{\rm T} J \left[P_r^{-1} H_0^{(\frac{1}{2},\dots,\frac{1}{2})} P_r\right] = zJ\,,$$

is indeed satisfied. The determinant of the permutation matrices is

$$\det P_r = (-\epsilon)^r\,. \tag{3.11}$$

[12] Similar symbols will be used below to indicate a symmetric or antisymmetric constant matrix.

Note that P_r is an element of G' iff $\det P_r = 1$.

The problem now is to find the transition functions among the 2^M patches just found. As in the case of $\mathrm{U}(N)$ vortices [25], the transition functions between the $(\frac{1}{2}, \ldots, \frac{1}{2})$-patch and the $(\underbrace{-\frac{1}{2}, \ldots, -\frac{1}{2}}_{r}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{M-r})$-patch are obtained by using the V-transformation (2.26):

$$H_0^{(\overbrace{-\frac{1}{2},\ldots,-\frac{1}{2}}^{r}, \overbrace{\frac{1}{2},\ldots,\frac{1}{2}}^{M-r})}(z; b'_{A,S}) = V(z) H_0^{(\frac{1}{2},\ldots,\frac{1}{2})}(z; b_{A,S}) \ . \tag{3.12}$$

By solving the above equation, one obtains the transition functions between the two patches having $\det P_r = 1$ as

$$b'_1 = \epsilon\, b_1^{-1} \,, \quad b'_2 = b_1^{-1} b_2 \,, \quad b'_3 = b_3 + \epsilon\, b_2^{\mathrm{T}} b_1^{-1} b_2 \,, \tag{3.13}$$

where $b_{A,S}$ is decomposed to an r-by-r matrix b_1, an r-by-$(M-r)$ matrix b_2 and an $(M-r)$-by-$(M-r)$ matrix b_3 defined as follows

$$b_{A,S} = \begin{pmatrix} b_1 & b_2 \\ -\epsilon\, b_2^{\mathrm{T}} & b_3 \end{pmatrix} , \quad b_{1,3}^{\mathrm{T}} = -\epsilon\, b_{1,3} \,, \tag{3.14}$$

and similarly for b'_i. The technical details will be postponed till the next section. This derivation of the quotient space $G'/\mathrm{U}(M)$ in the moduli matrix formalism, can be related to the ordinary derivation with $2M$ dimensional vector spaces which we call the orientation vectors. See appendix B for the details.

As shown in eq. (3.11), $\det P_r$ is always $+1$ in the case of $G' = \mathrm{USp}(2M)$, while both $+1$ and -1 are possible for $G' = \mathrm{SO}(2M)$. Hence, all 2^M patches can be connected for $G' = \mathrm{USp}(2M)$. However, two patches which are related by the permutation P_r with $\det P_r = -1$ are disconnected since such a permutation is not an element of $\mathrm{SO}(2M)$ but of $\mathrm{O}(2M)$ and thus there does not exist any transition function (V-transformation). Therefore, we conclude that the patches for $G' = \mathrm{SO}(2M)$ are divided into two disconnected parts according to the sign of $\det P_r = \pm 1$. In summary, the moduli space of the $k = 1$ vortex is

$$\mathcal{M}_{\mathrm{USp}(2M)} = \mathbb{C} \times \mathcal{M}^{\mathrm{ori}}_{\mathrm{USp}(2M)} = \mathbb{C} \times \frac{\mathrm{USp}(2M)}{\mathrm{U}(M)} \,, \tag{3.15}$$

$$\mathcal{M}_{\mathrm{SO}(2M)} = \mathbb{C} \times \mathcal{M}^{\mathrm{ori}}_{\mathrm{SO}(2M)} = \left(\mathbb{C} \times \frac{\mathrm{SO}(2M)}{\mathrm{U}(M)}\right)_+ \cup \left(\mathbb{C} \times \frac{\mathrm{SO}(2M)}{\mathrm{U}(M)}\right)_- \tag{3.16}$$

with $\mathbb{C}$ being the position moduli. The doubling of the moduli space in the $\mathrm{SO}(N)$ case reflects the presence of a $\mathbb{Z}_2$ topological charge for the vortex (see eq. (2.51)), so that $\mathcal{M}^{\mathrm{ori}}_{\mathrm{SO}(2M),+} \cap \mathcal{M}^{\mathrm{ori}}_{\mathrm{SO}(2M),-} = \emptyset$.

Furthermore, the structure of these moduli spaces seems to be consistent with the GNOW duality [43]. The dual of $\mathrm{USp}(2M)$ is the $\mathrm{Spin}(2M+1)$ group, with a single spinor representation of multiplicity, 2^M. In the case of $\mathrm{SO}(2M)$, its GNOW dual is $\mathrm{Spin}(2M)$, where the smallest irreducible representations are the two spinor representations of chirality $\pm$, each with multiplicity 2^{M-1}. Actually, the quotient $\mathrm{SO}(2M)/\mathrm{U}(M)$ is just a

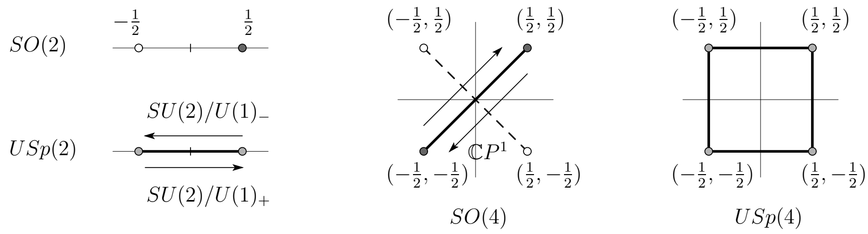

Figure 2. The moduli spaces of the $k = 1$ local vortex.

space for a pure spinor in $2M$ dimensions [50]. Finally, by embedding the vortex theory into an underlying theory with a larger gauge group which breaks to the group $\mathrm{SO}(2M)$ or to $\mathrm{USp}(2M)$, what is found here for the vortex moduli and their transformation properties can be translated into the properties of the monopoles appearing at the ends, through the homotopy matching argument [16, 26]. These aspects will be further discussed in a separate article [51].

We have introduced the dual weight diagram $\vec{\tilde{\mu}}$ to represent the special moduli matrices (representative vortex solutions), $H_0^{(\tilde{\mu}_1,\tilde{\mu}_2,\ldots,\tilde{\mu}_M)}(z)$ in section 2.2. Now we reinterpret them in a slightly different way. The lattice points of the diagram can be thought of as a representation of the patches of the space, where the origin of the local coordinates are just given by these special points. For example, in the case of $G' = \mathrm{SO}(2M), \mathrm{USp}(2M)$, the lattice point $\vec{\tilde{\mu}} = (\frac{1}{2}, \ldots, \frac{1}{2})$ represents the patch given in eq. (3.8).[13] Next we link the lattice points painted with the same color, namely the patches related by the permutation P_r with $\det P_r = +1$. The structure of the moduli space discussed above can easily be read off from the dual weight diagram obtained this way.

The dual lattices formed by special points representatives of connected patches are equal to lattices of irreducible representations of the dual group. On the contrary, two disconnected parts of the moduli space (see $\mathcal{M}_{\mathrm{SO}(2M)}$ in eq. (3.16)) nicely correspond to distinct irreducible representations (two spinor representations of opposite chiralities). In the case of composite vortices, we will find irreducible representations obtained by tensor compositions of the fundamental ones. This picture holds for all the explicit cases we could check (low rank groups), and is an important hint of a "semi-classical" emergence of the GNOW duality from the vortex side.

3.1.1 Examples: $G' = \mathrm{SO}(2), \mathrm{SO}(4), \mathrm{SO}(6)$ and $G' = \mathrm{USp}(2), \mathrm{USp}(4)$

Let us illustrate the structure of the moduli spaces in some simple cases, see figure 2. The $\mathrm{U}(1) \times \mathrm{SO}(2) \simeq \mathrm{U}(1)_+ \times \mathrm{U}(1)_-$ theory has two types of ANO vortices. One type is characterized by $\pi_1(\mathrm{U}(1)_+)$ and the other of $\pi_1(\mathrm{U}(1)_-)$. They are described by the following moduli matrices

$$H_0^{(\frac{1}{2})} = \begin{pmatrix} z - z_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_0^{(-\frac{1}{2})} = \begin{pmatrix} 1 & 0 \\ 0 & z - z_2 \end{pmatrix}. \tag{3.17}$$

[13]This interpretation gives an intrinsic meaning to the special points . Furthermore, their number is related (in many cases equal) to the Euler character of the moduli space.

Because $\mathrm{USp}(2) \simeq \mathrm{SU}(2)$, the $G' = \mathrm{USp}(2)$ vortex is indeed identical to the U(2) vortex which has been well-studied in the literature. The orientational moduli are $\mathbb{C}P^1 \simeq \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$. Note that the special configurations $H_0^{(-\frac{1}{2})} = \mathrm{diag}(1,z)$ and $H_0^{(\frac{1}{2})} = \mathrm{diag}(z,1)$ are fixed points of the $\mathrm{U}(1) \subset \mathrm{SU}(2)$ group generated by σ_3: $\mathrm{U}(1) = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. One can move from $H_0^{(-\frac{1}{2})}$ to $H_0^{(\frac{1}{2})}$ by using SU(2)/U(1) and vice versa [25]:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}}_{H_0^{(-\frac{1}{2})}} \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_{\mathrm{SU}(2)/\mathrm{U}(1)} = \underbrace{\begin{pmatrix} 0 & 1/a' \\ -a' & z \end{pmatrix}}_{V\text{-transformation}} \underbrace{\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}}_{H_0^{(\frac{1}{2})}} \underbrace{\begin{pmatrix} 1 & 0 \\ a' & 1 \end{pmatrix}}_{\mathrm{SU}(2)/\mathrm{U}(1)}, \quad \text{with} \quad aa' = 1\,. \tag{3.18}$$

The corresponding dual weight diagram, shown in the bottom-left of figure 2, represents the fundamental multiplet of the dual SU(2) group. It can be also interpreted as the toric diagram of $\mathbb{C}P^1$.

Next consider $G' = \mathrm{SO}(4)$ vortices. We have two different vortices which are characterized by the $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$-parity. The orientational moduli again turn out to be

$$\mathbb{C}P^1 \simeq \frac{\mathrm{SO}(4)}{\mathrm{U}(2)} \simeq \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathrm{U}(1) \times \mathrm{SU}(2)} \simeq \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\,. \tag{3.19}$$

For instance, we find a similar relation between $H_0^{(-\frac{1}{2},-\frac{1}{2})}$ and $H_0^{(\frac{1}{2},\frac{1}{2})}$

$$\underbrace{\begin{pmatrix} \mathbf{1}_2 & \\ & z\mathbf{1}_2 \end{pmatrix}}_{H_0^{(-\frac{1}{2},-\frac{1}{2})}} \underbrace{\begin{pmatrix} \mathbf{1}_2 & b_A \\ & \mathbf{1}_2 \end{pmatrix}}_{\mathrm{SO}(4)/\mathrm{U}(2)} = \underbrace{\begin{pmatrix} & {b'_A}^{-1} \\ -b'_A & z\mathbf{1}_2 \end{pmatrix}}_{V\text{-transformation}} \underbrace{\begin{pmatrix} z\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix}}_{H_0^{(\frac{1}{2},\frac{1}{2})}} \underbrace{\begin{pmatrix} \mathbf{1}_2 & \\ b'_A & \mathbf{1}_2 \end{pmatrix}}_{\mathrm{SO}(4)/\mathrm{U}(2)}, \quad \text{with } b_A b'_A = \mathbf{1}_2\,. \tag{3.20}$$

The two special points (the two sites of the dual weight diagram) are again fixed points of the U(1) symmetry, thus the dual weight diagram can be thought of as the toric diagram for $\mathbb{C}P^1$. There are two $\mathbb{C}P^1$'s in this case, see figure 2. Furthermore, the diagram can alternatively be thought of as representing the reducible $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \oplus (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$ representation of the spinor $Spin(4)$, which is the dual of SO(4).

The diagram for the $G' = \mathrm{USp}(4)$ case consists of a single structure where all the 4 points are connected

$$\mathcal{M}^{\mathrm{ori}}_{\mathrm{USp}(4)} = \frac{\mathrm{USp}(4)}{\mathrm{U}(2)}\,. \tag{3.21}$$

This is consistent with the interpretation of the diagram in figure 2 as being the weight lattice of the irreducible spinor representation **4** of SO(5), which is indeed the GNOW-dual of USp(4) [43].

The last example is $G' = \mathrm{SO}(6)$ (see figure 3). This is another neat example where the orientational moduli are a well-known manifold and its dual weight diagram can be identified with a toric diagram. The orientational moduli space is

$$\mathcal{M}^{\mathrm{ori}}_{\mathrm{SO}(6)} = \frac{\mathrm{SO}(6)}{\mathrm{U}(3)} \simeq \frac{\mathrm{SU}(4)}{\mathrm{U}(1) \times \mathrm{SU}(3)} \simeq \mathbb{C}P^3\,. \tag{3.22}$$

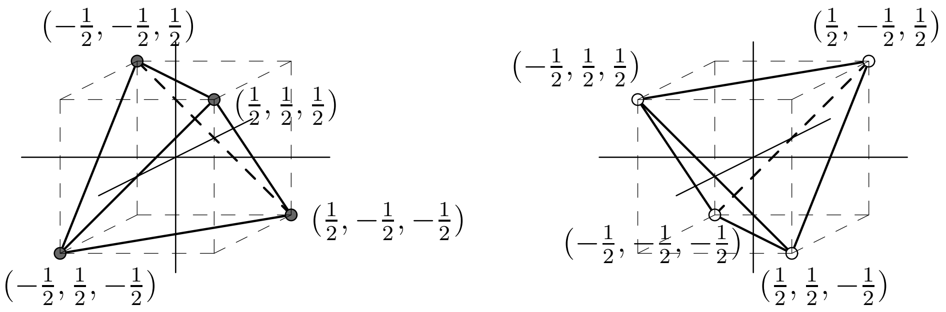

Figure 3. The moduli spaces of the $k = 1$ local vortex in $G' = \mathrm{SO}(6)$.

The corresponding dual weight diagram is shown in figure 3. There are two $\mathbb{C}P^3$'s similar to the case of $G' = \mathrm{SO}(4)$. From the toric diagram, one can easily find the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ subspaces which appear as edges and faces, respectively. Again these two separate parts of the moduli spaces can be interpreted as the two spinor representations, $\mathbf{4} \oplus \mathbf{4}^*$, of opposite chiralities of the dual group

$$\mathrm{Spin}(6) \sim \mathrm{SU}(4)\ .$$

3.2 The doubly-wound ($k = 2$) local vortex in $G' = \mathrm{SO}(2M)$ and $G' = \mathrm{USp}(2M)$ theories

In the case of $G' = \mathrm{SO}(2M), \mathrm{USp}(2M)$ theories, the strong condition for the $k = 2$ vortices located at $z = z_1$ and $z = z_2$ is of the form

$$H_0(z)^{\mathrm{T}} J H_0(z) = P(z) J\,, \quad P(z) \equiv (z - z_1)(z - z_2)\,, \tag{3.23}$$

which can equivalently be parametrized as

$$P(z) = (z - z_0)^2 - \delta\,, \quad z_0 = \frac{z_1 + z_2}{2}\,, \quad \delta = \left(\frac{z_1 - z_2}{2}\right)^2\ . \tag{3.24}$$

Here z_1 and z_2 stand for the vortex positions which are where the scalar field becomes zero, while z_0 and δ are the center of mass and the relative position (separation) of two vortices, respectively. Several examples of dual weight diagrams are given in figure 4.

We will now proceed to the doubly-wound ($k = 2$) vortices in $\mathrm{U}(1) \times G'$ gauge theories, with $G' = \mathrm{SO}(2M)$ or $\mathrm{USp}(2M)$. The $\mathrm{SU}(N)_{\mathrm{C+F}}$-orbit structure of the moduli space of k vortices in $\mathrm{U}(N)$ gauge theory was studied in ref. [27] using the Kähler quotient construction of Hanany and Tong [7]. Here we study the orbit structure of the moduli space of $k = 2$ vortices for $G' = \mathrm{SO}(2M)$ or $\mathrm{USp}(2M)$ more systematically by using the moduli matrix formalism. Before going into the detail, let us recall the properties of the $k = 2$ ANO vortices in the usual Abelian-Higgs model. They can be also studied using the moduli matrix which, in this case is simply a holomorphic function in z, i.e. a second-order polynomial:

$$H_0^{\mathrm{ANO}}(z) = z^2 - \alpha z + \beta = (z - z_1)(z - z_2)\,, \tag{3.25}$$

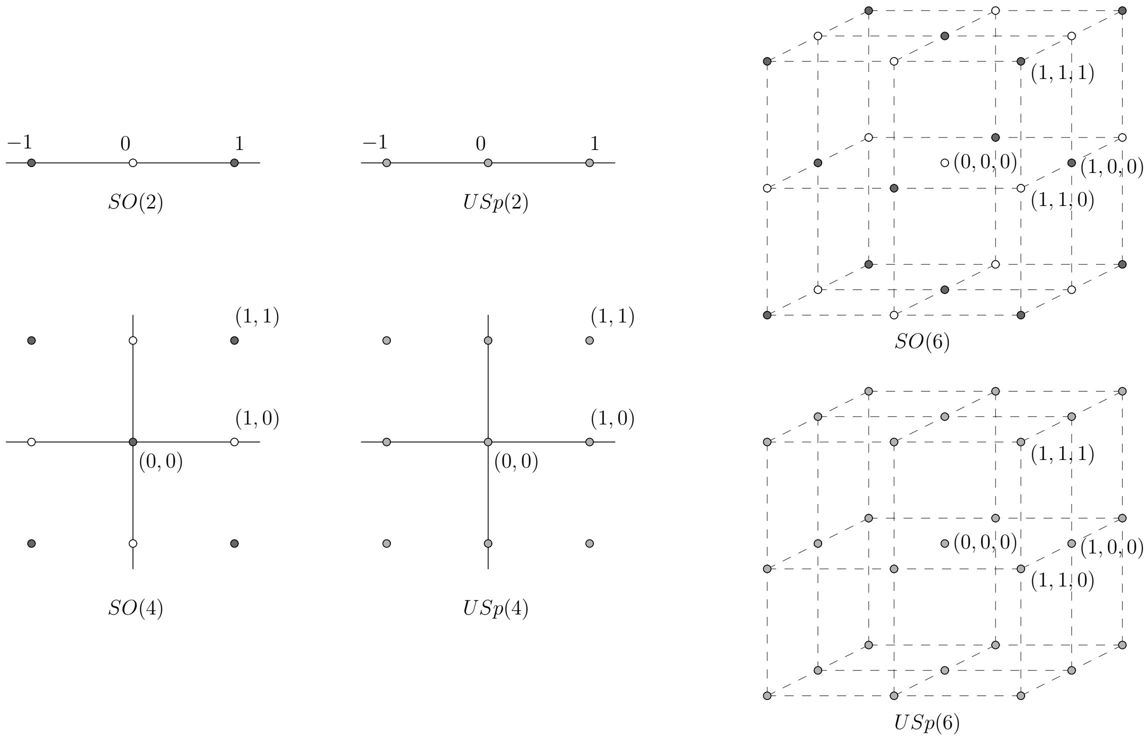

Figure 4. The special points for the $k = 2$ vortex.

with $\alpha = z_1 + z_2$ and $\beta = z_1 z_2$. Since these two vortices are indeed identical, we cannot distinguish them. In fact, the moduli matrix is invariant under the exchange of z_1 and z_2. Thus the corresponding moduli space is the symmetric product of $\mathbb{C}$:

$$\mathcal{M}_{\rm ANO}^{k=2} = \frac{\mathbb{C} \times \mathbb{C}}{\mathfrak{S}_2} \simeq \mathbb{C}^2/\mathbb{Z}_2 \ . \tag{3.26}$$

There is a nice property of the moduli matrix for the local vortices. Suppose H_0^i satisfies the strong condition for k_i local vortices, namely $(H_0^i)^{\rm T} J H_0^i = P_i(z) J$ with a polynomial function of the k_i-th power. Then the product of two matrices $H_0^{(i,j)} \equiv H_0^i H_0^j$ automatically satisfies the strong condition for $k = k_i + k_j$ local vortices: $(H_0^{(i,j)})^{\rm T} J H_0^{(i,j)} = P_i(z) P_j(z) J$. In this way we can construct the moduli matrices for the higher winding number vortices from those with the lower winding numbers, which was found in U(N) vortices [15, 16]. This feature implies that the moduli space for separated local vortices can be constructed as a symmetric product of copies of those of a single local vortex:

$$\mathcal{M}_{\rm sep}^{k} \simeq \frac{(\mathbb{C} \times \mathcal{M}_{\rm ori})^k}{\mathfrak{S}_k} \tag{3.27}$$

The consideration above is valid when the component vortices are separated even for small vortex separations. When two or more vortex axes coalesce, the symmetric product degenerates, and the topological structure of the moduli space undergoes a change. Thus the coincident case must be treated more carefully. We shall study the case of two coincident vortices in detail in the next section.

Our study of the moduli matrix in the present work is complete up to $k = 2$ vortices ($k = 1$ for odd SO groups). The problem of a complete classification of the moduli matrix for the higher winding number ($k \geq 3$) is left for future work.

The product of moduli matrices, especially for the $G' = \mathrm{SO}(N)$ case, gives us a natural understanding in the following sense. The single $G' = \mathrm{SO}(N)$ vortex has a $\mathbb{Z}_2$-parity $+1$ or -1. They are physically distinct, hence the $k = 2$ configuration is expected to be classified into three categories by the $\mathbb{Z}_2$-parity of the component vortex as $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, +1), (+1, -1), (-1, -1)$. The total $\mathbb{Z}_2$-parity of the configurations with $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, +1), (-1, -1)$ is $+1$ while that of $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, -1)$ is -1. Therefore, the former and the latter are disconnected. An interesting question is whether $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, +1)$ and $(-1, -1)$ are connected or not. The naive answer would be yes, because the two solutions represent two equivalent objects from the topological point of view. However, the true answer, as we will show, is subtler, and is different for the local and semi-local cases. For the latter case, the two moduli spaces are smoothly connected and in fact are the same space. More interestingly, in the local case they represent two different spaces which intersect at some submanifold. As we shall see, this result is compatible with the interpretation that weight lattices formed by connected special points are in correspondence with irreducible representations of the dual group [21].[14]

The patch structure for the $k = 2$ local vortices in generic $G' = \mathrm{SO}(2M), \mathrm{USp}(2M)$ theories is rather complex. In this subsection, we just present the result without details. The result will be discussed again when we shall consider the generic configurations satisfying the weak condition (2.22) in section 4. The moduli matrix in a generic patch takes the form

$$H_0^{(\overbrace{1,\ldots,1}^{r},\overbrace{0,\ldots,0}^{M-r})}(z) = \begin{pmatrix} P(z)\mathbf{1}_r & 0 & 0 & 0 \\ B_1(z) & (z-z_0)\mathbf{1}_{M-r} + \Gamma_{11} & 0 & \Gamma_{12} \\ A(z) & C_1 & \mathbf{1}_r & C_2 \\ B_2(z) & \Gamma_{21} & 0 & (z-z_0)\mathbf{1}_{M-r} + \Gamma_{22} \end{pmatrix}, \tag{3.28}$$

$$A(z) = a_{1;A,S}\, z + a_{0;A,S} + \lambda_{S,A}\,, \tag{3.29}$$

$$\begin{pmatrix} B_1(z) \\ B_2(z) \end{pmatrix} = -\left((z-z_0)\mathbf{1}_{2(M-r)} + \Gamma\right) J_{2(M-r)} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}, \tag{3.30}$$

$$\Gamma \equiv \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}, \tag{3.31}$$

where $a_{i;A,S}$ $(i = 0, 1)$ is an $r \times r$ constant (anti-)symmetric matrix, C_i is an $r \times (M - r)$ constant matrix and we have defined

$$\lambda_{S,A} \equiv -\frac{1}{2}(C_1,\ C_2)\, J_{2(M-r)} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}, \quad J_{2(M-r)} \equiv \begin{pmatrix} & \mathbf{1}_{M-r} \\ \epsilon\,\mathbf{1}_{M-r} & \end{pmatrix}. \tag{3.32}$$

[14]The fact that there is no topology which can explain this disconnection somehow enforces our interpretation in terms of the dual group.

The strong condition is now translated into the following form

$$\Gamma^{\mathrm{T}} J_{2(M-r)} + J_{2(M-r)}\Gamma = 0\,, \quad \Gamma^2 = \delta\,\mathbf{1}_{2(M-r)}\,, \qquad (\mathrm{Tr}\,\Gamma = 0)\ . \tag{3.33}$$

Solutions to this condition for separated vortices are discussed in appendix C.1. It is a hard task to study the moduli space collecting all the patches, for generic SO($2M$) and USp($2M$). A complete analysis of the moduli space in several cases will be given later.

Some of the moduli parameters in eq. (3.28) are the Nambu-Goldstone (NG) modes associated with global symmetry breaking and the rest are interpreted as so-called quasi-NG modes [52]. The former is, for instance, the overall orientation of the two vortices and the center of mass. The relative separation between two local vortices ($\mathbb{C}$) and some of the relative orientational modes are typical examples of the latter. For two coincident vortices the situation is subtler, but in general there will still be a set of NG modes generated by the $G'_{\mathrm{C+F}}$ symmetry, while the remaining modes are quasi-NG modes. As we will see in the following, the number of the quasi-NG modes is $\left[\frac{M}{2}\right]$ or $\left[\frac{M}{2}\right]-1$ for SO($2M$) and M for USp($2M$), which was actually difficult to find without using the moduli matrix formalism.

3.2.1 $G'_{\mathrm{C+F}}$-orbits for coincident vortices

Let us now specialize to the case of the $k=2$ co-axial (axially symmetric) vortices. The details of the analysis can be found in appendix C.2. Consider a special moduli matrix

$$H_0^{(\overbrace{1,\ldots,1}^{r},\overbrace{0,\ldots,0}^{M-r})} = \mathrm{diag}\big(\underbrace{z^2,\ldots,z^2}_{r},\underbrace{z,\ldots,z}_{M-r},\underbrace{1,\ldots,1}_{r},\underbrace{z,\ldots,z}_{M-r}\big)\ . \tag{3.34}$$

Clearly, this vortex breaks the color-flavor symmetry as

$$\mathrm{SO}(2M) \to \mathrm{U}(r)\times\mathrm{SO}(2(M-r))\,, \quad \mathrm{USp}(2M) \to \mathrm{U}(r)\times\mathrm{USp}(2(M-r))\ . \tag{3.35}$$

Thus depending on r ($r=0,1,\ldots,M$), we have $M+1$ different $G'_{\mathrm{C+F}}$ orbits. Each orbit reflects the NG modes associated with the symmetry breaking. The different orbits are connected by the quasi-NG modes which are unrelated to symmetry. The total space is stratified with $G'_{\mathrm{C+F}}$-orbits as leaves. To see this, let us consider the following moduli matrix (for $G' = \mathrm{SO}(2M)$):

$$H_0 = \begin{pmatrix} z^2\mathbf{1}_{M-2} & & & \\ & z\mathbf{1}_2 & & i\sigma_2\lambda \\ & & \mathbf{1}_{M-2} & \\ & & & z\mathbf{1}_2 \end{pmatrix} = V^{-1}\begin{pmatrix} z^2\mathbf{1}_{M-2} & & & \\ & z^2\mathbf{1}_2 & & \\ & & \mathbf{1}_{M-2} & \\ & -i\sigma_2\lambda^{-1}z & & \mathbf{1}_2 \end{pmatrix}, \tag{3.36}$$

$$V = \begin{pmatrix} \mathbf{1}_{M-2} & & & \\ & z\mathbf{1}_2 & & -i\sigma_2\lambda \\ & & \mathbf{1}_{M-2} & \\ & -i\sigma_2\lambda^{-1} & & \mathbf{0}_2 \end{pmatrix} \in \mathrm{SO}(2M)\ . \tag{3.37}$$

We can always take λ to be non-negative and real $\mathbb{R}_{>0}$ by means of the color-flavor rotation

$$H_0 \to U^{-1} H_0 U \,, \quad U = \begin{pmatrix} \mathbf{1}_{M-2} & & & \\ & a\mathbf{1}_2 & & \\ & & \mathbf{1}_{M-2} & \\ & & & a^{-1}\mathbf{1}_2 \end{pmatrix} \in \mathrm{SO}(2M) \,. \tag{3.38}$$

In two limits $\lambda \to 0$ and $\lambda \to \infty$, the moduli matrix (3.36) reduces to the special matrix (3.34) with $r = M-2$ and $r = M$, respectively. The orbit with intermediate values $0 < \lambda < \infty$ corresponds to the symmetry breaking pattern

$$\frac{\mathrm{SO}(2M)}{\mathrm{U}(M-2) \times \mathrm{USp}(2)} \,. \tag{3.39}$$

In fact, the moduli matrix (3.36) is left invariant under the $\mathrm{USp}(2) \in \mathrm{SO}(2M)_{\mathrm{C+F}}$ transformations

$$U = \begin{pmatrix} \mathbf{1}_{M-2} & & & \\ & g^{-1} & & \\ & & \mathbf{1}_{M-2} & \\ & & & g^{\mathrm{T}} \end{pmatrix} \in \mathrm{SO}(2M) \,, \quad g^{\mathrm{T}}(i\sigma_2)g = i\sigma_2 \,. \tag{3.40}$$

Therefore, the quasi-NG mode λ connects two different $\mathrm{SO}(2M)_{\mathrm{C+F}}$ orbits:

$$\frac{\mathrm{SO}(2M)}{\mathrm{U}(M)} \times \mathbb{Z}_2 \overset{\lambda\to 0}{\longleftarrow} \mathbb{R}_{>0} \times \frac{\mathrm{SO}(2M)}{\mathrm{U}(M-2)\times\mathrm{USp}(2)} \times \mathbb{Z}_2 \overset{\lambda\to\infty}{\longrightarrow} \frac{\mathrm{SO}(2M)}{\mathrm{U}(M-2)\times\mathrm{SO}(4)} \,, \tag{3.41}$$

where the $\mathbb{Z}_2$ factor indicates a permutation, $P^{-1}H_0P$ with $P \in \mathrm{O}(2M)/\mathrm{SO}(2M)$. This permutation does not belong to the $\mathrm{SO}(2M)_{\mathrm{C+F}}$ symmetry, nonetheless it generates a new moduli matrix solution. We thus see, as explained before, how the moduli space of coincident vortices of positive chirality is generically made of two disconnected parts. If $M - r \neq 0$, such a permutation acts trivially or can be pulled back by an $\mathrm{SO}(2M)$ rotation on H_0. At these special points the two copies coalesce. Nonetheless we must interpret the two spaces as defining two different composite states of vortices: $(+1,+1)$ and a $(-1,-1)$. This interpretation is fully consistent if one studies interactions in the range of validity of the moduli space approximation [58]. It is easy to realize that, in this approximation, the chirality of each of the component vortices is conserved: two composite states of vortices $(+1,+1)$ and $(-1,-1)$ do not interact, even if their trajectories in the moduli space pass through an intersection submanifold[15]

At the intersection, the dimension of the manifold always reduces by

$$[\dim \mathbb{R}_{>0} - \dim \mathrm{USp}(2)] - (-\dim \mathrm{SO}(4)) = 4 \,.$$

[15] The question if (or how) these vortices interact beyond the moduli space approximation, and in particular at the quantum level, is an interesting open question. See also a comment related to this issue at the end of the Conclusion.

This can easily be extended to the following moduli matrix, with $t, \alpha \in \mathbb{Z}_{\geq 0}$

$$H_0 = \left(\begin{array}{ccc|ccc} z^2 \mathbf{1}_t & & & & & \\ & z^2 \mathbf{1}_{2\alpha} & & & & \\ & & z \mathbf{1}_{M-t-2\alpha} & & & \\ \hline 0 & & & \mathbf{1}_t & & \\ & z\tilde{\Lambda} & & & \mathbf{1}_{2\alpha} & \\ & & 0 & & & z\mathbf{1}_{M-t-2\alpha} \end{array}\right), \quad \tilde{\Lambda} = \begin{pmatrix} \tilde{\lambda}_1 \tilde{J}_{2\tilde{p}_1} & & \\ & \ddots & \\ & & \tilde{\lambda}_s \tilde{J}_{2\tilde{p}_s} \end{pmatrix}, \tag{3.42}$$

where $\tilde{J}_{2\tilde{p}_i}$ is the invariant tensor of $\mathrm{USp}(2\tilde{p}_i)$ and

$$\alpha = \sum_{i=1}^{s} \tilde{p}_i \,, \quad t + 2\alpha \leq M \,, \quad 0 < \tilde{\lambda}_i < \tilde{\lambda}_{i+1} \;. \tag{3.43}$$

An arbitrary patch (3.28) with $\delta = 0$ in the $\mathrm{SO}(2M)$ case, can be brought onto the above form as explained in appendix C.2. The set of numbers $(t, s, \tilde{p}_i)$ and the quasi-NG modes λ_i are, of course, independent of r which indicates the patch which we take as a starting point.

Note that this is invariant with respect to the group $\prod_{i=1}^{s} \mathrm{USp}(2\tilde{p}_i) \in \mathrm{SO}(2M)_{\mathrm{C+F}}$

$$U = \text{block-diag}\left(\mathbf{1}_t, g_{2\tilde{p}_1}^{-1}, \ldots, g_{2\tilde{p}_s}^{-1}, \mathbf{1}_{M-t-2\alpha}, \mathbf{1}_t, g_{2\tilde{p}_1}^{\mathrm{T}}, \ldots, g_{2\tilde{p}_s}^{\mathrm{T}}, \mathbf{1}_{M-t-2\alpha}\right) , \tag{3.44}$$

with $g_{2\tilde{p}_i}^{\mathrm{T}} \tilde{J}_{2\tilde{p}_i} g_{2\tilde{p}_i} = \tilde{J}_{2\tilde{p}_i}$. Therefore, the local structure of the $\mathrm{SO}(2M)$-orbit has the form

$$\mathbb{R}_{>0}^{s} \times \frac{\mathrm{O}(2M)}{\mathrm{U}(t) \times \prod_{i=1}^{s} \mathrm{USp}(2\tilde{p}_i) \times \mathrm{O}(2u)} \,, \quad \text{with} \quad t + u + 2\sum_{i=1}^{s} \tilde{p}_i = M \;. \tag{3.45}$$

When we take the limit $\tilde{\lambda}_1 \to 0$, a subgroup $\mathrm{U}(t) \times \mathrm{USp}(2\tilde{p}_1)$ of the isotropy group gets enhanced to $\mathrm{U}(t+2\tilde{p}_1)$ and the orbit shrinks, thus the local structure around the new orbit is given by changing the indices in eq. (3.45) as follows

$$(s, t, u; \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_s) \quad \overset{\tilde{\lambda}_1 \to 0}{\to} \quad (s', t', u'; \tilde{p}'_i) = (s-1, t+2\tilde{p}_1, u; \tilde{p}_2, \ldots, \tilde{p}_s) \;. \tag{3.46}$$

In the opposite limit where $\tilde{\lambda}_s \to \infty$, another subgroup $\mathrm{USp}(2\tilde{p}_s) \times \mathrm{SO}(2u)$ of the isotropy group is getting enlarged to $\mathrm{SO}(2u + 4\tilde{p}_s)$, hence the local structure around this new orbit is obtained by

$$(s, t, u; \tilde{p}_1, \ldots, \tilde{p}_{s-1}, \tilde{p}_s) \quad \overset{\tilde{\lambda}_s \to \infty}{\to} \quad (s'', t'', u''; \tilde{p}''_i) = (s-1, t, u+2\tilde{p}_s; \tilde{p}_1, \ldots, \tilde{p}_{s-1}). \tag{3.47}$$

By choosing various $t, \tilde{p}_i$ and taking the limits $\tilde{\lambda}_i \to 0, \infty$, we can reach all the points of the moduli space. However, since these transitions are always induced by the $2\tilde{p}_i \times 2\tilde{p}_i$ matrix $\tilde{J}_{2\tilde{p}_i}$, the patches with only an even number of z^2's in the diagonal element are connected. Analogously, the patches with an odd number of z^2's are mutually connected. Nevertheless, the former and latter remain disconnected and this of course is just a consequence of the different chiralities ($\mathbb{Z}_2$ topological factor).

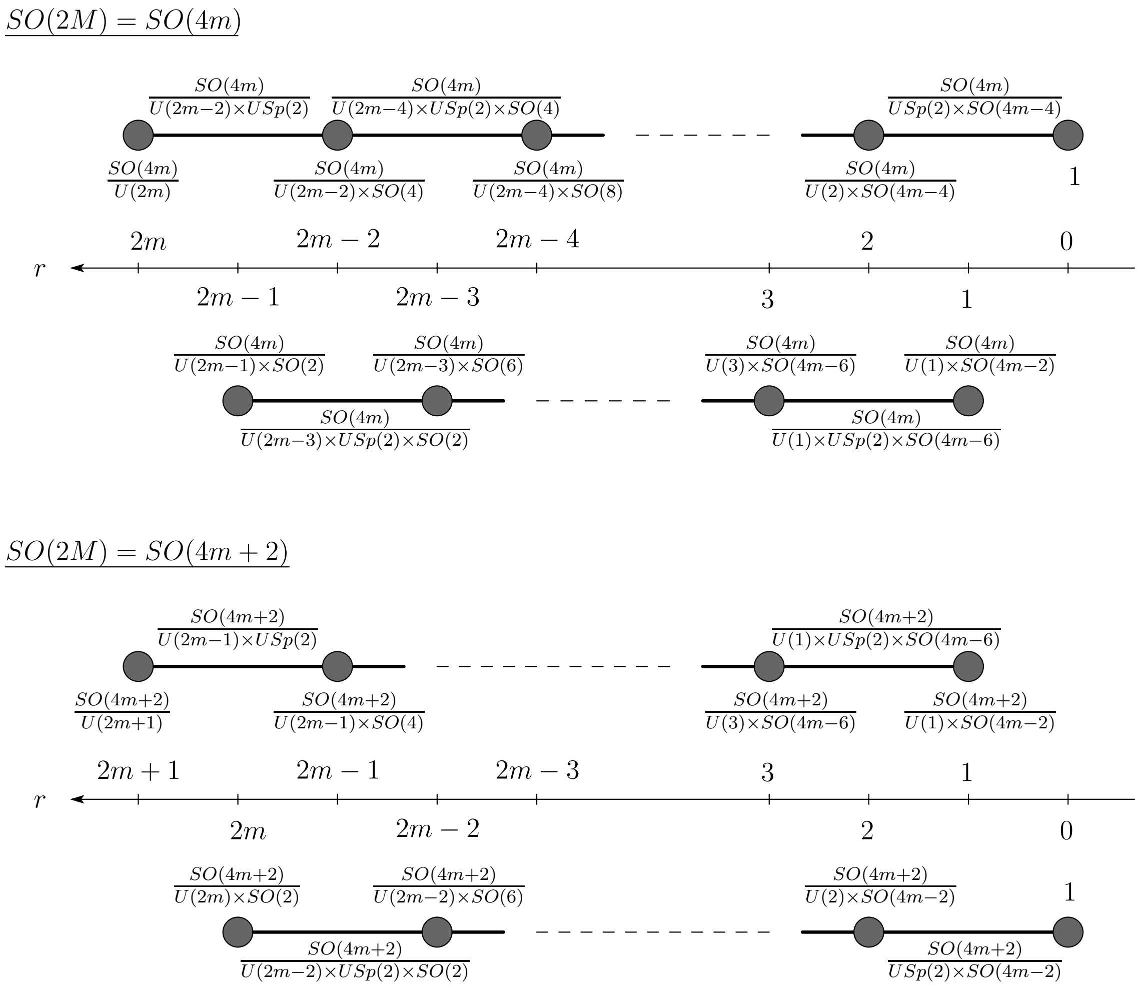

Figure 5. Sequences of the $k=2$ vortices in $\mathrm{SO}(4m)$ and $\mathrm{SO}(4m+2)$. The sites (circles) correspond to the special orbits of eq. (3.36) and the links connecting them denote the insertion of the minimal pieces $\tilde{\lambda}_i \tilde{J}_2$ such as in eq. (3.42).

For instance, by inserting a minimal extension, i.e. the following piece, $\tilde{\lambda}\tilde{J}_2$, the special orbits in eq. (3.34) can sequentially be shifted as

$$\begin{aligned}\mathrm{diag}(z^2,\ldots,z^2,z^2,z^2,1,\ldots,1,1,1) &\to \mathrm{diag}(z^2,\ldots,z^2,z,z,1,\ldots,1,z,z) \to \cdots \\ &\to \mathrm{diag}(z,\ldots,z,z\cdots,z)\ .\end{aligned}$$

However, the connection pattern depends on whether $\mathrm{SO}(2M)=\mathrm{SO}(4m)$ or $\mathrm{SO}(4m+2)$, see figure 5. At a generic point ($\tilde{p}_i = 1,\ s=m$) where the color-flavor symmetry is maximally broken the corresponding moduli spaces can locally be written as

$$\mathcal{M}^{k=2,\,\mathrm{ori}}_{\mathrm{SO}(4m),+} = \mathbb{R}^m_{>0} \times \frac{\mathrm{SO}(4m)}{\mathrm{USp}(2)^m} \times \mathbb{Z}_2\,, \tag{3.48}$$

$$\mathcal{M}^{k=2,\,\mathrm{ori}}_{\mathrm{SO}(4m),-} = \mathbb{R}^{m-1}_{>0} \times \frac{\mathrm{SO}(4m)}{\mathrm{U}(1)\times\mathrm{USp}(2)^{m-1}\times\mathrm{SO}(2)}\,, \tag{3.49}$$

$$
\mathcal{M}^{k=2,\,\mathrm{ori}}_{\mathrm{SO}(4m+2),+} = \mathbb{R}^m_{>0} \times \frac{\mathrm{SO}(4m+2)}{\mathrm{U}(1)\times\mathrm{USp}(2)^m} \times \mathbb{Z}_2\,, \tag{3.50}
$$

$$
\mathcal{M}^{k=2,\,\mathrm{ori}}_{\mathrm{SO}(4m+2),-} = \mathbb{R}^m_{>0} \times \frac{\mathrm{SO}(4m+2)}{\mathrm{USp}(2)^m\times\mathrm{SO}(2)}\;. \tag{3.51}
$$

The two copies of the moduli space, in the case of positive chirality, intersect at some submanifold if $M \neq 1$. The dimensions of these moduli spaces are summarized as

$$
\dim_{\mathbb{C}}\left[\mathcal{M}^{k=2,\,\mathrm{ori}}_{\mathrm{SO}(2M),\pm}\right] = M^2 - M\;. \tag{3.52}
$$

Taking the vortex position into account, the complex dimension of the full moduli space is $M^2 - M + 2$ which is nothing but twice the dimension of the $k = 1$ moduli space.

In the case of vortices in $\mathrm{USp}(2M)$ theory, we can bring a generic moduli matrix onto the following form

$$
H_0 = \left(\begin{array}{ccc|ccc}
z^2\mathbf{1}_t & & & & & \\
 & z^2\mathbf{1}_\beta & & & & \\
 & & z\mathbf{1}_{M-t-\beta} & & & \\
\hline
0 & & & \mathbf{1}_t & & \\
 & z\tilde{\Lambda} & & & \mathbf{1}_\beta & \\
 & & 0 & & & z\mathbf{1}_{M-t-\beta}
\end{array}\right), \quad
\tilde{\Lambda} = \begin{pmatrix}
\tilde{\lambda}_1\mathbf{1}_{\tilde{p}_1} & & \\
 & \ddots & \\
 & & \tilde{\lambda}_s\mathbf{1}_{\tilde{p}_s}
\end{pmatrix} \tag{3.53}
$$

with

$$
\beta = \sum_{i=1}^{s}\tilde{p}_i, \quad t+\beta \leq M, \quad 0 < \tilde{\lambda}_i < \tilde{\lambda}_{i+1}. \tag{3.54}
$$

This matrix is invariant under $[\prod_{i=1}^{s}\mathrm{O}(\tilde{p}_i)] \in \mathrm{USp}(2M)$

$$
U = \text{block-diag}\left(\mathbf{1}_t, g^{-1}_{\tilde{p}_1}, \ldots, g^{-1}_{\tilde{p}_s}, \mathbf{1}_{M-t-\beta}, \mathbf{1}_t, g^{\mathrm{T}}_{\tilde{p}_1}, \ldots, g^{\mathrm{T}}_{\tilde{p}_s}, \mathbf{1}_{M-t-\beta}\right), \tag{3.55}
$$

with $g^{\mathrm{T}}_{\tilde{p}_i} g_{\tilde{p}_i} = \mathbf{1}_{\tilde{p}_i}$. Therefore, the local structure around the $\mathrm{USp}(2M)$ orbit is given by

$$
\mathbb{R}^s_{>0} \times \frac{\mathrm{USp}(2M)}{\mathrm{U}(t)\times[\prod_{i=1}^{s}\mathrm{O}(\tilde{p}_i)]\times\mathrm{USp}(2u)}, \quad \text{with} \quad t+u+\sum_{i=1}^{s}\tilde{p}_i = M\;. \tag{3.56}
$$

In the limit $\tilde{\lambda}_1 \to 0$, the local structures of the orbit changes according to

$$
(s,t,u;\tilde{p}_1,\tilde{p}_2,\ldots,\tilde{p}_s) \overset{\tilde{\lambda}_1\to 0}{\to} (s',t',u';\tilde{p}'_i) = (s-1,t+\tilde{p}_1,u;\tilde{p}_2,\ldots,\tilde{p}_s)\;. \tag{3.57}
$$

On the other hand, in the opposite limit $\tilde{\lambda}_s \to \infty$, the local structure of the orbit becomes

$$
(s,t,u;\tilde{p}_1,\ldots,\tilde{p}_{s-1},\tilde{p}_s) \overset{\tilde{\lambda}_s\to\infty}{\to} (s'',t'',u'';\tilde{p}''_i) = (s-1,t,u+\tilde{p}_s;\tilde{p}_1,\ldots,\tilde{p}_{s-1})\;. \tag{3.58}
$$

Since the minimal insertion is a real positive number $\tilde{\lambda}$, all the special orbits are connected, contrary to the case of the $\mathrm{SO}(2M)$ vortices. This is consistent with the fact that there is no $\mathbb{Z}_2$-parity in the $\mathrm{USp}(2M)$ case.

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Figure 6. The $k = 2$ local vortices for $G' = \mathrm{SO}(2), \mathrm{USp}(2)$.

At the most generic point where $0 < \tilde{\lambda}_1 < \cdots < \tilde{\lambda}_M$, the color-flavor symmetry is broken down to the discrete subgroup $\mathbb{Z}_2^M$,

$$\mathbb{R}^M_{>0} \times \frac{\mathrm{USp}(2M)}{\mathbb{Z}_2^M} \, . \tag{3.59}$$

We can read off the dimensions of moduli space for the $k = 2$ co-axial local $\mathrm{USp}(2M)$ vortices from this

$$\dim_{\mathbb{C}} \left[\mathcal{M}^{k=2,\mathrm{ori}}_{\mathrm{USp}(2M)} \right] = \frac{M}{2} + \frac{2M(2M+1)}{4} = M^2 + M \, . \tag{3.60}$$

3.2.2 Examples: $G' = \mathrm{SO}(2), \mathrm{SO}(4)$ and $G' = \mathrm{USp}(2), \mathrm{USp}(4)$

$k = 2$ local vortices for $G' = \mathrm{SO}(2), \mathrm{USp}(2)$

Let us first consider the $G' = \mathrm{SO}(2)$ theory. Although there is no $\mathbb{Z}_2$-parity due to the fact that $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$, there are nevertheless two distinct classes of vortices characterized by $\pi_1(\mathrm{U}(1)_+)$ and $\pi_1(\mathrm{U}(1)_-)$ with $\mathrm{U}(1) \times \mathrm{SO}(2) \simeq \mathrm{U}(1)_+ \times \mathrm{U}(1)_-$. Thus there are three possible $k = 2$ configurations. $(\pi_1(\mathrm{U}(1)_+), \pi_1(\mathrm{U}(1)_-)) = \{(2,0), (0,2), (1,1)\}$, see figure 6. The corresponding moduli matrices are given by

$$H_0^{(+1)} = \begin{pmatrix} P(z) & 0 \\ 0 & 1 \end{pmatrix} \, , \quad H_0^{(-1)} = \begin{pmatrix} 1 & 0 \\ 0 & P(z) \end{pmatrix} \, , \quad H_0^{(0)} = \begin{pmatrix} z - z_1 & 0 \\ 0 & z - z_2 \end{pmatrix} \, . \tag{3.61}$$

Clearly, z_1 and z_2 are not distinguishable in the first two matrices while they are in the third matrix. This reflects the fact that the configuration consists of two identical vortices and two different vortices, in the two respective cases. Therefore, the moduli space is made of three disconnected pieces

$$\mathcal{M}^{k=2}_{\mathrm{SO}(2)} = \mathcal{M}^{(2,0)}_{\mathrm{SO}(2)} \cup \mathcal{M}^{(0,2)}_{\mathrm{SO}(2)} \cup \mathcal{M}^{(1,1)}_{\mathrm{SO}(2)} \, , \tag{3.62}$$

where these spaces are defined by

$$\mathcal{M}^{(2,0)}_{\mathrm{SO}(2)} = \left(\mathcal{M}^{(1,0)}_{\mathrm{SO}(2)} \times \mathcal{M}^{(1,0)}_{\mathrm{SO}(2)} \right) / \mathfrak{S}_2 = (\mathbb{C} \times \mathbb{C}) / \mathfrak{S}_2 = \mathbb{C}^2 / \mathbb{Z}_2 \, , \tag{3.63}$$

$$\mathcal{M}^{(0,2)}_{\mathrm{SO}(2)} = \left(\mathcal{M}^{(0,1)}_{\mathrm{SO}(2)} \times \mathcal{M}^{(0,1)}_{\mathrm{SO}(2)} \right) / \mathfrak{S}_2 = (\mathbb{C} \times \mathbb{C}) / \mathfrak{S}_2 = \mathbb{C}^2 / \mathbb{Z}_2 \, , \tag{3.64}$$

$$\mathcal{M}^{(1,1)}_{\mathrm{SO}(2)} = \mathcal{M}^{(1,0)}_{\mathrm{SO}(2)} \times \mathcal{M}^{(0,1)}_{\mathrm{SO}(2)} = \mathbb{C}^2 \, . \tag{3.65}$$

The $\mathbb{Z}_2$ factor gives rise to crucial differences in the interactions between these vortices. For instance, a head-on collision of two identical local vortices in $\mathcal{M}^{(2,0)}_{\mathrm{SO}(2)}$ or $\mathcal{M}^{(0,2)}_{\mathrm{SO}(2)}$ leads

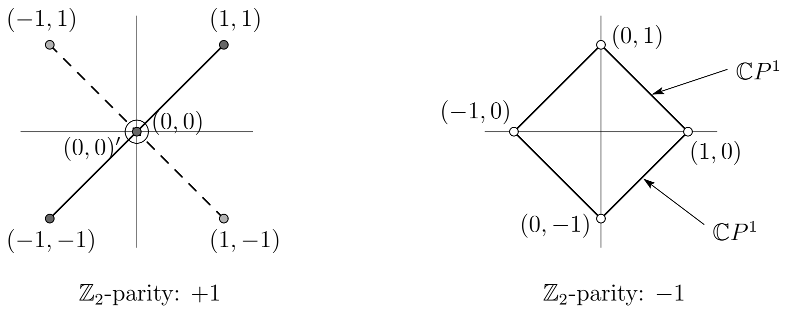

Figure 7. The patches of the $k = 2$ local vortices in $G' = \mathrm{SO}(4)$.

to a 90 degree scattering, while such a collision of the two different local vortices living in $\mathcal{M}^{(1,1)}_{\mathrm{SO}(2)}$ would be transparent, which yields opposite results for the reconnection of two colliding vortex-strings [28]. Again, this result is a consequence of the fact that vortices with different chiralities must be considered as different, and non-interacting objects.

The next example is $G' = \mathrm{USp}(2)$. As was noted earlier the vortices in the $G' = \mathrm{USp}(2)$ theory are the ones thoroughly studied due to $\mathrm{USp}(2) = \mathrm{SU}(2)$. The moduli spaces including the patches and the transition functions for the $k = 2$ vortices, in terms of the moduli matrix, are given in ref. [15, 16]. We shall not repeat the discussion here. The result is [15, 16]

$$\begin{aligned} \mathcal{M}^{k=2,\text{separated}}_{\mathrm{SU}(2)} &\simeq (\mathbb{C}\times\mathbb{C}P^1)^2/\mathfrak{S}_2, \\ \mathcal{M}^{k=2,\text{coincident}}_{\mathrm{SU}(2)} &\simeq \mathbb{C}\times W\mathbb{C}P^2_{(2,1,1)} \simeq \mathbb{C}\times\mathbb{C}P^2/\mathbb{Z}_2 \,. \end{aligned} \tag{3.66}$$

The dual weight diagram for this case is shown in figure 6.

$k = 2$ local vortices for $G' = \mathrm{SO}(4)$

Let us now consider $G' = \mathrm{SO}(4)$. As can be seen from figure 4, there are 9 special points in the entire moduli space. Five out of them have $Q_{\mathbb{Z}_2} = +1$, and the other four have $Q_{\mathbb{Z}_2} = -1$.

Note that the isomorphism $\mathrm{SO}(4) \simeq [\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-]/\mathbb{Z}_2$ can indeed be complexified as

$$\begin{aligned} \mathrm{SO}(4)^{\mathbb{C}} &\simeq [\mathrm{SL}(2,\mathbb{C})_+ \times \mathrm{SL}(2,\mathbb{C})_-]/\mathbb{Z}_2\,, \\ [\mathrm{U}(1)\times\mathrm{SO}(4)]^{\mathbb{C}}/\mathbb{Z}_2 &\simeq [\mathrm{GL}(2,\mathbb{C})_+ \times \mathrm{GL}(2,\mathbb{C})_-]/\mathbb{C}^* \,. \end{aligned} \tag{3.67}$$

In fact, an arbitrary matrix X satisfying $X^{\mathrm{T}}JX \propto J$ can always be rewritten as

$$\begin{aligned} X &= \sigma^{-1}(A\otimes B)\sigma = f_+(A)f_-(B) = f_-(B)f_+(A), \\ f_+(A) &= \sigma^{-1}(A\otimes\mathbf{1}_2)\sigma, \quad f_-(B) = \sigma^{-1}(\mathbf{1}_2\otimes B)\sigma, \quad \sigma = \begin{pmatrix} \mathbf{1}_2 & & \\ & & 1 \\ & -1 & \end{pmatrix} \end{aligned} \tag{3.68}$$

where $A, B \in \mathrm{GL}(2,\mathbb{C})$ and $f_\pm$ define maps from $\mathrm{GL}(2,\mathbb{C})_\pm$ to $[\mathrm{U}(1)\times\mathrm{SO}(4)]^{\mathbb{C}}/\mathbb{Z}_2$. The elements of $\mathrm{GL}(2,\mathbb{C})_\pm$, $f_\pm(A)$, are related by the odd parity permutation

$$P^{-1} f_\pm(A) P = f_\mp(A)\,, \quad P = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix}, \quad (\det P = -1)\,. \tag{3.69}$$

Fixed points of this permutation are given by $A \propto \mathbf{1}_2$. This complexified isomorphism tells us that a moduli matrix for $G' = \mathrm{SO}(4)$ obeying the strong condition can always be decomposed to a couple of the moduli matrices for $G' = \mathrm{SU}(2)$ which have been well-studied. This fact simplifies the analysis of the moduli space in the present case. For instance, $f_\pm$ are maps from the moduli matrix for $k = 1, G' = \mathrm{SU}(2)$ to those of $k = 1, \mathrm{SO}(4)$ with the parity $Q_{\mathbb{Z}_2} = \pm 1$, since $f_+(\mathrm{diag}(z,1)) = \mathrm{diag}(z,z,1,1)$.

Consider first the $Q_{\mathbb{Z}_2} = +1$ patches. There are corresponding patches of the four special points $\vec{\mu} = (\pm 1, \pm 1), (\pm 1, \mp 1)$. The $(1,1)$-patch is explicitly given by the moduli matrix

$$H_0^{(1,1)} = \begin{pmatrix} z^2 + b_1 z + b_2 & & & \\ & z^2 + b_1 z + b_2 & & \\ & -b_3 z - b_4 & 1 & \\ b_3 z + b_4 & & & 1 \end{pmatrix}, \tag{3.70}$$

with $(z - z_1)(z - z_2) = z^2 + b_1 z + b_2$. The rest of the patches $H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}$ can be obtained by appropriate permutations of $H_0^{(1,1)}$. Note that the special point $\vec{\mu} = (0,0)$ of the moduli space has two different vicinities which we call the $(0,0)_+$-patch and the $(0,0)_-$-patch, that is, the point $\vec{\mu} = (0,0)$ is on an intersection of two submanifolds. In fact, we find that the two different matrices

$$H_0^{(0,0)+} = \begin{pmatrix} z - a_1 & & & a_4 \\ & z - a_1 & -a_4 & \\ & a_3 & z - a_2 & \\ -a_3 & & & z - a_2 \end{pmatrix}, \quad H_0^{(0,0)-} = \begin{pmatrix} z - a'_1 & a'_4 & & \\ -a'_3 & z - a'_2 & & \\ & & z - a'_2 & a'_3 \\ & & -a'_4 & z - a'_1 \end{pmatrix}, \tag{3.71}$$

with

$$(z - z_1)(z - z_2) = (z - a_1)(z - a_2) + a_3 a_4 = (z - a'_1)(z - a'_2) + a'_3 a'_4\,, \tag{3.72}$$

are connected at the points where $a_3 = a_4 = a'_3 = a'_4 = 0$ and $a_1 = a_2 = a'_1 = a'_2$ only. As mentioned, these concrete expressions for the patches can be obtained by the maps from those of the $G' = \mathrm{SU}(2)$ case as follows

$$H_0^{(0,0)+} = f_+(h^{(1,1)}(a_i))\,, \quad H_0^{(1,1)} = f_+(h^{(2,0)}(b_i))\,, \quad H_0^{(-1,-1)} = f_+(h^{(0,2)}(c_i))\,, \tag{3.73}$$

$$H_0^{(0,0)-} = f_-(h^{(1,1)}(a'_i))\,, \quad H_0^{(1,-1)} = f_-(h^{(2,0)}(b'_i))\,, \quad H_0^{(-1,1)} = f_-(h^{(0,2)}(c'_i))\,, \tag{3.74}$$

where $h^{(*,*)}(a_i)$ are the moduli matrices for $G' = \mathrm{SU}(2), k = 2$,

$$h^{(1,1)}(a_i) = \begin{pmatrix} z - a_1 & a_4 \\ -a_3 & z - a_2 \end{pmatrix} ,$$
$$h^{(2,0)}(b_i) = \begin{pmatrix} z^2 + b_1 z + b_2 & 0 \\ b_3 z + b_4 & 1 \end{pmatrix} , \quad h^{(0,2)}(c_i) = \begin{pmatrix} 1 & c_3 z + c_4 \\ 0 & z^2 + c_1 z + c_2 \end{pmatrix} . \tag{3.75}$$

The transition functions among these patches are given by the V-transformation (2.26) with $V(z) = f_+(V_+(z)) f_-(V_-(z))$ where $V_\pm(z)$ are those of $G' = \mathrm{SU}(2)$, i.e. they are exactly the same as in the SU(2) case [10, 11, 15]. Now, connectedness of the patches is manifest since we know the moduli space for $G' = \mathrm{SU}(2)$ is indeed simply connected. The three patches in eq. (3.73) compose a submanifold $\mathcal{M}^{k=2}_{\mathrm{SO}(4),++}$ and eq. (3.74) composes $\mathcal{M}^{k=2}_{\mathrm{SO}(4),--}$. The moduli space with $Q_{\mathbb{Z}_2} = +1$, therefore, can be expressed as

$$\mathcal{M}^{k=2}_{\mathrm{SO}(4),+} \simeq \mathcal{M}^{k=2}_{\mathrm{SO}(4),++} \cup \mathcal{M}^{k=2}_{\mathrm{SO}(4),--} , \quad \mathcal{M}^{k=2}_{\mathrm{SO}(4),++} \simeq \mathcal{M}^{k=2}_{\mathrm{SO}(4),--} \simeq \mathcal{M}^{k=2}_{\mathrm{SU}(2)} , \tag{3.76}$$

where $\mathcal{M}^{k=2}_{\mathrm{SU}(2)}$ is shown in eq. (3.66). As we have shown, these two submanifolds intersect at the fixed point of the permutation (3.69) in the $(0,0)_+$-patch and the $(0,0)_-$-patch

$$\mathcal{M}^{k=2}_{\mathrm{SO}(4),++} \cap \mathcal{M}^{k=2}_{\mathrm{SO}(4),--} = \mathbb{C} , \tag{3.77}$$

where $\mathbb{C}$ describes the position of the two coincident local vortices, $a_1 = a_2 = a_1' = a_2'$. Note that by comparing the right panel of figure 6 and the left panel of figure 7 (with a ± 45 degrees rotation), it is easily seen that the $k = 2$, U(2) moduli spaces are embedded in that of the SO(4) theory.

Let us next study the transition functions among the $Q_{\mathbb{Z}_2} = -1$ patches, $(1,0)$-$(0,1)$-$(-1,0)$-$(0,-1)$. The general form of the moduli matrix in the $(1,0)$-patch is:

$$H_0^{(1,0)} = f_+(h^{(1,0)}(z_1, d_1)) f_-(h^{(1,0)}(z_2, d_2)) = \begin{pmatrix} (z - z_1)(z - z_2) & & & \\ -d_2(z - z_1) & z - z_1 & & \\ -d_1 d_2 & d_1 & 1 & d_2 \\ -d_1(z - z_2) & & & z - z_2 \end{pmatrix} , \tag{3.78}$$

while the other three are

$$\begin{aligned} H_0^{(0,1)} &= f_+(h^{(1,0)}(z_1, d_1)) f_-(h^{(0,1)}(z_2, e_2)) , \\ H_0^{(0,-1)} &= f_+(h^{(0,1)}(z_1, e_1)) f_-(h^{(1,0)}(z_2, d_2)) , \\ H_0^{(-1,0)} &= f_+(h^{(0,1)}(z_1, e_1)) f_-(h^{(0,1)}(z_2, e_2)) , \end{aligned} \tag{3.79}$$

where $h^{(1,0)}$ and $h^{(0,1)}$ are the two patches of $\mathcal{M}^{k=1}_{\mathrm{SU}(2)} \simeq \mathbb{C} \times \mathbb{C}P^1$,

$$h^{(1,0)}(z_0, b) = \begin{pmatrix} z - z_0 & \\ -b & 1 \end{pmatrix} , \quad h^{(0,1)}(z_0, b') = \begin{pmatrix} 1 & -b' \\ & z - z_0 \end{pmatrix} . \tag{3.80}$$

Hence, we can conclude that the moduli space of the $k = 2$ local vortices with $Q_{\mathbb{Z}_2} = -1$ is

$$\mathcal{M}^{k=2}_{\mathrm{SO}(4),-} \simeq (\mathcal{M}^{k=1}_{\mathrm{SU}(2)})^2 \simeq \left(\mathbb{C} \times \mathbb{C}P^1\right)^2 . \tag{3.81}$$

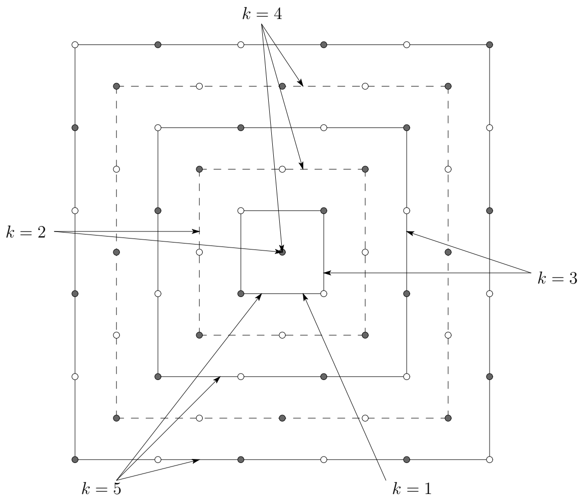

Figure 8. The dual weight lattice for $k = 1, 2, 3, 4, 5$ vortices in $G' = \mathrm{SO}(4)$.

This can be also understood from the dual weight diagrams in figures 2 and 7.

The difference between the moduli spaces in eq. (3.76) and eq. (3.81) can be understood as follows. Recall that there exist two kinds of minimal vortices in $G' = \mathrm{SO}(2M)$ theory, namely one for $\mathrm{SU}(2)_+$ with $Q_{\mathbb{Z}_2} = +1$ and another for $\mathrm{SU}(2)_-$ with $Q_{\mathbb{Z}_2} = -1$, see figure 2. We can then choose two vortices with either the same or a different $\mathbb{Z}_2$-parity in composing the $k = 2$ vortex. Two vortices with the same parity can be regarded as physically identical, while those with different parities are distinct. In the case of two identical vortices, the moduli space should be a symmetric product, namely given by eq. (3.76). Since the total parity $Q^{k=2}_{\mathbb{Z}_2} = +1$ can be made of $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, +1)$ and $(-1, -1)$, one finds two copies, as in eq. (3.76). In contrast, there is only one possibility for $Q^{k=2}_{\mathbb{Z}_2} = -1$, namely $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, -1)$. The dual weight diagrams are thus quite useful. As a further illustration, let us show the diagrams for some higher-winding vortices with $G' = \mathrm{SO}(4)$ in figure 8, without going into any detail.

$k = 2$ local vortices for $G' = \mathrm{USp}(4)$

Consider now the $k = 2$ local vortices for $G' = \mathrm{USp}(4)$. Since the moduli for a single ($k = 1$) local vortex requires four parameters, we expect that the $k = 2$ configurations need eight. The moduli matrices including the special points as the origin are of the form

$$H_0^{(0,0)} = (z - z_0)\mathbf{1}_4 + A\,, \tag{3.82}$$

$$H_0^{(1,0)} = \begin{pmatrix} P(z) & 0 & 0 & 0 \\ b_3 b_6 - b_4(z - z_0 + b_5) & z - z_0 + b_5 & 0 & b_6 \\ b_1 z + b_2 & b_3 & 1 & b_4 \\ -b_4 b_7 + b_3(z - z_0 - b_5) & b_7 & 0 & z - z_0 - b_5 \end{pmatrix}, \tag{3.83}$$

with $P(z) = (z - z_0)^2 - \delta = (z - z_0)^2 - (b_5^2 + b_6 b_7)$ and

$$H_0^{(1,1)} = \begin{pmatrix} P(z) & 0 & 0 & 0 \\ 0 & P(z) & 0 & 0 \\ c_3 z + c_4 & c_5 z + c_6 & 1 & 0 \\ c_5 z + c_6 & c_7 z + c_8 & 0 & 1 \end{pmatrix} , \tag{3.84}$$

where $P(z) = z^2 + c_1 z + c_2$. All other patches are connected and can be obtained by suitable permutations. The moduli matrices $H_0^{(1,1)}, H^{(1,0)}$ depend on eight free parameters, as expected. The strong condition is already solved by them, and thus these patches are $\mathbb{C}^8$. The moduli matrix $H_0^{(0,0)}$ has however a more complicated form. The strong condition turns out to be:

$$A^{\mathrm{T}} J + JA = 0 , \quad A^2 = \delta \mathbf{1}_4 \ . \tag{3.85}$$

The first condition tells that A takes a value in the algebra of USp(4), so

$$A = \begin{pmatrix} -\frac{a_{12}-a_{34}}{2} & a_{35} & a_{13} & a_{15} \\ -a_{45} & -\frac{a_{12}+a_{34}}{2} & a_{15} & -a_{14} \\ a_{24} & a_{25} & \frac{a_{12}-a_{34}}{2} & a_{45} \\ a_{25} & -a_{23} & -a_{35} & \frac{a_{12}+a_{34}}{2} \end{pmatrix} . \tag{3.86}$$

Now A has 10 parameters. The second set of constraints comes from imposing the Plücker condition on $a_{ij} = -a_{ji}$ $(i, j = 1, 2, 3, 4, 5)$

$$a_{ij} a_{kl} - a_{ik} a_{jl} + a_{il} a_{jk} = 0 \ . \tag{3.87}$$

Note that the number of linearly independent conditions is three, hence seven parameters out of ten in the matrix A are linearly independent. Those together with z_0, yield eight degrees of freedom, indeed as expected. In this patch, δ depends on a_{ij} as follows

$$\delta = \frac{1}{4} (a_{12} - a_{34})^2 + a_{13} a_{24} - a_{35} a_{45} + a_{15} a_{25} \ . \tag{3.88}$$

Thus the patch $H^{(0,0)}$ is expressed as

$$\begin{aligned} \{H_0^{(0,0)}\} &\simeq \mathbb{C} \times \frac{\{B | B : 2 \times 5 \text{ matrix}\}}{\mathrm{SL}(2,\mathbb{C})} \simeq \mathbb{C} \times \left(\mathbb{C}^* \rtimes \frac{\{B | B : 2 \times 5 \text{ matrix}\}}{\mathrm{GL}(2,\mathbb{C})} \right) \simeq \\ &\simeq \mathbb{C} \times \left(\mathbb{C} \rtimes \frac{\{B | B : 2 \times 5 \text{ matrix of rank } 2\}}{\mathrm{GL}(2,\mathbb{C})} \right) = \mathbb{C} \times (\mathbb{C} \rtimes Gr_{5,2}) \ . \end{aligned} \tag{3.89}$$

The last term in the bracket is a cone whose base space is a U(1) fibration of $Gr_{5,2}$. The tip of this cone corresponds to the origin of the patch, where $a_{ij} = 0$, which is thus a conical singularity in the moduli space. Notice that this is a true singularity of the classical metric on the moduli space. It comes out by applying the strong condition on a smooth set of coordinates [28]. It is an interesting open problem how this singularity affects the interactions of vortices. The transition functions between these patches are

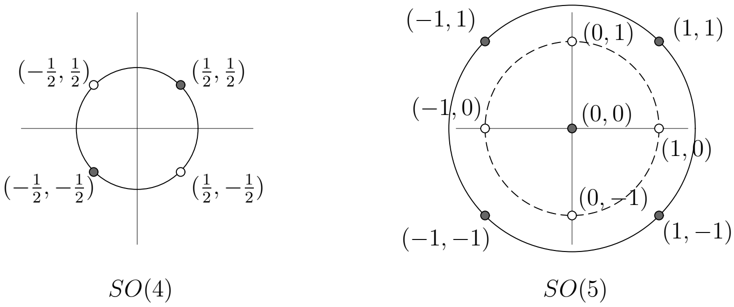

Figure 9. Comparison between the single (minimum-winding) vortices in $G' = \mathrm{SO}(4)$ and $G' = \mathrm{SO}(5)$ theories.

easily obtained, for instance, by requiring that $V(z) = H^{(1,1)}(H^{1,0})^{-1}$ be regular with respect to z

$$c_1 = -2z_0, \quad c_2 = z_0^2 - b_5^2 - b_6 b_7, \qquad c_3 = b_1 + \frac{b_4^2}{b_6}, \quad c_4 = b_2 - \frac{1}{b_6}(b_3 b_4 b_6 - b_4^2(b_5 - z_0)),$$
$$c_5 = -\frac{b_4}{b_6}, \quad c_6 = b_3 - \frac{b_4}{b_6}(b_5 - z_0), \quad c_7 = \frac{1}{b_6}, \qquad c_8 = \frac{1}{b_6}(b_5 - z_0) \ . \tag{3.90}$$

The parameters in $H^{(1,0)}$ are transformed to $a_{ij} = B_{1i}B_{2j} - B_{2i}B_{1j}$ of $H^{(0,0)}$ as

$$B \simeq \frac{1}{\sqrt{b_1}} \begin{pmatrix} 1 & b_3^2 - b_1 b_7 & 0 & -b_2 - z_0 b_1 + b_3 b_4 + b_1 b_5 & -b_3 \\ 0 & -b_2 - z_0 b_1 - b_3 b_4 - b_1 b_5 & 1 & -b_4^2 - b_1 b_6 & b_4 \end{pmatrix} . \tag{3.91}$$

3.3 The $k = 1$ local vortex for $G' = \mathrm{SO}(2M + 1)$

Let us now consider the vortex solutions of the $G' = \mathrm{SO}(2M + 1)$ theory. The strong condition for the $k = 1$ local vortex positioned at the origin in $G' = \mathrm{SO}(2M + 1)$ is given by eq. (3.1) with $n_0 = 1$. It is very similar to the condition eq. (3.23) for the $k = 2$ coincident vortices ($z_1 = z_2 = 0$) in $G' = \mathrm{SO}(2M)$

$$H_0^{\mathrm{T}} J H_0 = z^2 J \ . \tag{3.92}$$

This implies that the complexity of a single local $\mathrm{SO}(2M+1)$ vortex is almost the same as in the case of the $k = 2$ co-axial $\mathrm{SO}(2M)$ vortices. Indeed, the corresponding dual weight diagrams, see figures 1 and 4, for instance, are the same.

If however we restrict ourselves to the case of the minimal vortex, there is a startling difference between the case of $\mathrm{SO}(2M)$ and that of $\mathrm{SO}(2M+1)$. Consider the dual weight diagrams in these two types of theories. In the case of the $\mathrm{SO}(2M)$ theory, all the weight vectors have the same length $|\tilde{\vec{\mu}}|^2 = M/4$, whereas those for the $\mathrm{SO}(2M+1)$ local vortices have different lengths $|\tilde{\vec{\mu}}|^2$ from 0 to M, see figure 9 for SO(4) and SO(5). The $M - 1$ dimensional sphere represents an orbit of $G'_{\mathrm{C+F}} = \mathrm{SO}(2M)$ or $G'_{\mathrm{C+F}} = \mathrm{SO}(2M+1)$ which is nothing but the internal orientation moduli. In the case of $G' = \mathrm{SO}(2M)$, the single vortex has only one orbit, hence the moduli space consists of the position $\mathbb{C}$ and the broken color-flavor symmetry $\mathrm{SO}(2M)/\mathrm{U}(M)$. On the other hand, in the case of $G' = \mathrm{SO}(2M+1)$, there exist multiple orbits corresponding to the NG modes, and furthermore the quasi-NG modes connecting them. For concreteness, let us consider the following moduli matrix

$$H_0^{(\overbrace{1,\dots,1}^{r},\overbrace{0,\dots,0}^{M-r})}(z) = \mathrm{diag}\big(\underbrace{z^2,\dots,z^2}_{r},\underbrace{z,\dots,z}_{M-r},\underbrace{1,\dots,1}_{r},\underbrace{z,\dots,z}_{M-r},z\big)\,, \tag{3.93}$$

where r takes on integer values from 0 to M. We now act with the color-flavor symmetry $G'_{\mathrm{C+F}} = \mathrm{SO}(2M+1)$ on the moduli matrix from the right. Hence, the $\mathrm{U}(r)$ subgroup in $\mathrm{SO}(2M+1)$ can be absorbed by the V-transformation (2.26):

$$U_0 = \begin{pmatrix} g^{-1} & & & & \\ & \mathbf{1}_{M-r} & & & \\ & & g^{\mathrm{T}} & & \\ & & & \mathbf{1}_{M-r} & \\ & & & & 1 \end{pmatrix} \in \mathrm{U}(r) \subset \mathrm{SO}(2M+1)\,, \qquad g \in \mathrm{U}(r)\,. \tag{3.94}$$

The other subgroup $\mathrm{SO}(2(M-r)+1) \subset \mathrm{SO}(2M+1)$ can be also absorbed by a V-transformation. Thus the orbit including the special point (3.93) is [21]

$$\mathcal{M}^r_{\mathrm{ori}} = \frac{\mathrm{SO}(2M+1)}{\mathrm{U}(r)\times \mathrm{SO}(2(M-r)+1)}\,. \tag{3.95}$$

The orbit continuously connects the special points corresponding to the dual weight vectors of the same lengths, see figure 9. Although the internal moduli spaces (3.95) with different r's are not connected by the action of $\mathrm{SO}(2M+1)$; these are indeed connected by quasi-NG modes.

The complete moduli space for the $k=1$, $\mathrm{SO}(2M+1)$ vortex is very similar to that of $k=2$ co-axial $\mathrm{SO}(2M)$ vortices which have been studied in section 3.2.1. A generic solution to the strong condition (3.92) is given by

$$H_0^{(\overbrace{1,\dots,1}^{r},\overbrace{0,\dots,0}^{M-r})}(z) = \begin{pmatrix} (z-z_0)^2\mathbf{1}_r & 0 & 0 & 0 \\ B_1(z) & (z-z_0)\mathbf{1}_{M-r}+\Gamma_{11} & 0 & \Gamma_{12} \\ A(z) & C_1 & \mathbf{1}_r & C_2 \\ B_2(z) & \Gamma_{21} & 0 & (z-z_0)\mathbf{1}_{M-r+1}+\Gamma_{22} \end{pmatrix}, \tag{3.96}$$

$$A(z) \equiv a_{1;A}\,z + a_{0;A} + \lambda_S\,, \tag{3.97}$$

$$\begin{pmatrix} B_1(z) \\ B_2(z) \end{pmatrix} = -\big((z-z_0)\mathbf{1}_{2(M-r)+1} + \Gamma\big)\, J_{2(M-r)+1} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}, \tag{3.98}$$

$$\Gamma \equiv \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}, \tag{3.99}$$

where $a_{i;A}$ $(i=0,1)$ are $r\times r$ constant anti-symmetric matrices, C_1 is an $r\times(M-r)$ constant matrix, C_2 is an $r\times(M-r+1)$ constant matrix, and we have defined

$$\lambda_S \equiv -\frac{1}{2}(C_1,\ C_2)\, J_{2(M-r)+1} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}, \qquad J_{2(M-r)+1} \equiv \begin{pmatrix} & \mathbf{1}_{M-r} & \\ \mathbf{1}_{M-r} & & \\ & & 1 \end{pmatrix}. \tag{3.100}$$

The strong condition is now translated into the following form

$$\Gamma^{\mathrm{T}} J_{2(M-r)+1} + J_{2(M-r)+1}\Gamma = 0\,, \quad \Gamma^2 = 0\,. \tag{3.101}$$

All moduli parameters are included in $a_{i;A}, C_i, \Gamma$. As in the case of $k = 2$ co-axial $G' = \mathrm{SO}(2M)$ vortices (see appendix C.2), $a_{0;A}$ and C_i can be removed by an appropriate color-flavor rotation and Γ satisfying the strong condition (3.101) can be written as (up to $\mathrm{SO}(2M+1)_{\mathrm{C+F}}$ rotations)

$$\Gamma \simeq \left(\begin{array}{cc|cc|c} & & \Lambda & & \\ & & & \mathbf{0}_{M-r-2\gamma} & \\ \hline \mathbf{0}_{2\gamma} & & & & \\ & \mathbf{0}_{M-r-2\gamma} & & & \\ \hline & & & & 0 \end{array}\right), \quad \Lambda \equiv i\sigma_2 \otimes \mathrm{diag}\left(\lambda_1 \mathbf{1}_{p_1}, \ldots, \lambda_q \mathbf{1}_{p_q}\right)\,, \tag{3.102}$$

with $\lambda_i > \lambda_{i+1} > 0$ and 2γ ($< 2(M-r)+1$) being the rank of Γ ($\gamma = \sum_{i=1}^{q} p_i$). By making use of the V-transformation and the $\mathrm{SO}(2M+1)_{\mathrm{C+F}}$ symmetry, we finally obtain the following moduli matrix

$$H_0 = \left(\begin{array}{cc|cc||cc|cc||c} z^2\mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2\mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & z^2\mathbf{1}_{2\gamma} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 \\ 0 & \Lambda'\,z & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 & 0 \\ \hline 0 & 0 & \Lambda^{-1}z & 0 & 0 & 0 & \mathbf{1}_{2\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 \\ \hline\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{array}\right), \tag{3.103}$$

where we have diagonalized $a_{1;A}$ as

$$a_{1;A} = u\Lambda' u^{\mathrm{T}}\,, \quad \Lambda' \equiv i\sigma_2 \otimes \mathrm{diag}\left(\lambda'_1 \mathbf{1}_{p'_1}, \ldots, \lambda'_{q'} \mathbf{1}_{p'_{q'}}\right)\,, \quad u \in \mathrm{U}(2\alpha)\,, \tag{3.104}$$

with 2α being the rank of $a_{1;A}$ and $2\alpha = 2\sum_{i=1}^{q'} p'_i$. Let us now rearrange the eigenvalues $\{\lambda_i^{-1}, \lambda'_i\}$ as

$$\mathrm{diag}(\Lambda', \Lambda^{-1}) \to i\sigma_2 \otimes \mathrm{diag}\left(\tilde{\lambda}_1 \mathbf{1}_{\tilde{p}_1}, \ldots, \tilde{\lambda}_s \mathbf{1}_{\tilde{p}_s}\right)\,, \quad \tilde{\lambda}_a > \tilde{\lambda}_{a+1} > 0\,, \tag{3.105}$$

and redefine $t \equiv r - 2\alpha,\ u \equiv M - r - 2\gamma$ with the constraint:

$$s, t, u \in \mathbb{Z}_{\geq 0}\,, \quad \tilde{p}_i \in \mathbb{Z}_{>0}\,, \quad t + u + 2\sum_{i=1}^{s} \tilde{p}_i = M\,, \tag{3.106}$$

such that the r-dependence in the form of eq. (3.103) disappears. We conclude that the moduli space of vortices is (apart from the center of mass position):

$$\mathcal{M}^{k=1,\mathrm{ori}}_{\mathrm{SO}(2M+1)} = \bigcup_{\{t,u,\tilde{p}_i |\, \mathrm{eq.}\ (3.106)\}} \mathbb{R}^s_{>0} \times \mathcal{O}_{t,u,\tilde{p}_i}\,, \tag{3.107}$$

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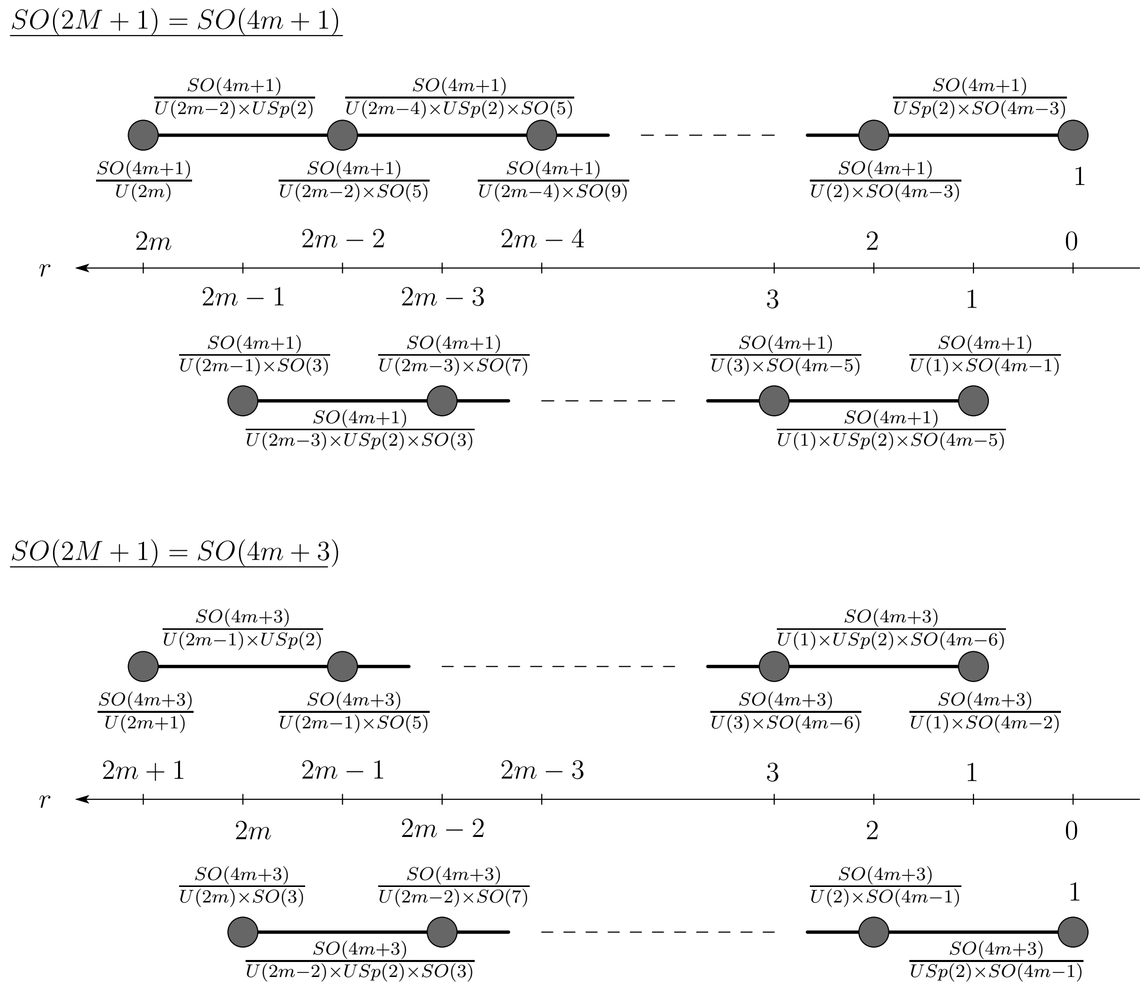

Figure 10. Sequences of the $k = 1$ vortices for $\mathrm{SO}(4m+1)$ and for $\mathrm{SO}(4m+3)$ theories.

$$\mathcal{O}_{t,u,\tilde{p}_i} = \frac{\mathrm{SO}(2M+1)}{\mathrm{U}(t) \times \mathrm{SO}(2u+1) \times \prod_{a=1}^{s} \mathrm{USp}(2\tilde{p}_a)} . \tag{3.108}$$

Note that there does not appear any $\mathbb{Z}_2$ factor contrary to the $\mathrm{SO}(2M)$ case since

$$P = \mathrm{diag}(1, \ldots, 1, -1) \in \mathrm{O}(2M+1)/\mathrm{SO}(2M+1)$$

acts trivially on H_0 in eq. (3.103). The special orbits in eq. (3.95) are obtained simply by choosing $s = 0$. A sequence of the moduli space is given in figure 10. At the most generic points, the moduli spaces are locally of the form

$$\mathcal{M}^{k=1,\mathrm{ori}}_{\mathrm{SO}(4m+1),+} = \mathbb{R}^m_{>0} \times \frac{\mathrm{SO}(4m+1)}{\mathrm{USp}(2)^m} , \tag{3.109}$$

$$\mathcal{M}^{k=1,\mathrm{ori}}_{\mathrm{SO}(4m+1),-} = \mathbb{R}^{m-1}_{>0} \times \frac{\mathrm{SO}(4m+1)}{\mathrm{U}(1) \times \mathrm{USp}(2)^{m-1} \times \mathrm{SO}(3)} , \tag{3.110}$$

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$$\mathcal{M}^{k=1,\mathrm{ori}}_{\mathrm{SO}(4m+3),+} = \mathbb{R}^m_{>0} \times \frac{\mathrm{SO}(4m+3)}{\mathrm{U}(1)\times \mathrm{USp}(2)^m}\,, \tag{3.111}$$

$$\mathcal{M}^{k=1,\mathrm{ori}}_{\mathrm{SO}(4m+3),-} = \mathbb{R}^m_{>0} \times \frac{\mathrm{SO}(4m+3)}{\mathrm{USp}(2)^m\times \mathrm{SO}(3)}\,. \tag{3.112}$$

The dimensions of the moduli spaces are then summarized as

$$\dim_{\mathbb{C}}\left[\mathcal{M}^{k=1,\mathrm{ori}}_{\mathrm{SO}(2M+1),+}\right] = M^2\,, \tag{3.113}$$

$$\dim_{\mathbb{C}}\left[\mathcal{M}^{k=1,\mathrm{ori}}_{\mathrm{SO}(2M+1),-}\right] = M^2-1\,. \tag{3.114}$$

3.3.1 Examples: $G' = \mathrm{SO}(3), \mathrm{SO}(5)$

$k=1$ local vortex for $G' = \mathrm{SO}(3)$

Let us discuss the simplest example, viz. $G' = \mathrm{SO}(3)$. In this model there are two patches having $Q_{\mathbb{Z}_2} = +1$. The moduli matrices take the respective forms

$$H_0^{(1)} = f_3(h^{(1,0)}(0,a)) = \begin{pmatrix} z^2 & 0 & 0 \\ -a^2 & 1 & \sqrt{2}a \\ -\sqrt{2}az & 0 & z \end{pmatrix}\,, \quad H_0^{(-1)} = f_3(h^{(0,1)}(0,b))\,. \tag{3.115}$$

where $h^{(*,*)}(z_0,a)$ are the two patches (3.80) of $\mathcal{M}^{k=1}_{\mathrm{SU}(2)}$ and the map f_3 is defined by

$$f_3 : A = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in GL(2,\mathbb{C}) \to f_3(A) = \begin{pmatrix} c^2 & -d^2 & \sqrt{2}cd \\ -e^2 & f^2 & -\sqrt{2}ef \\ \sqrt{2}ce & -\sqrt{2}df & cf+de \end{pmatrix}\,, \tag{3.116}$$

and expresses the isomorphism $GL(2,\mathbb{C})/\mathbb{Z}_2 \simeq [\mathrm{U}(1)\times \mathrm{SO}(3)]^{\mathbb{C}}$. On the other hand, there exists just a single patch with $Q_{\mathbb{Z}_2} = -1$. This "patch" actually contains only a point

$$H_0^{(0)} = f_3(\sqrt{z}\mathbf{1}_2) = z\mathbf{1}_3\,. \tag{3.117}$$

This vortex does not break the color-flavor symmetry $G'_{\mathrm{C+F}} = \mathrm{SO}(3)$: it is an Abelian vortex i.e. not having any orientational moduli. Hence, the moduli spaces $\mathcal{M}^{k=1}_{\mathrm{SO}(3),\pm}$ are

$$\mathcal{M}^{k=1}_{\mathrm{SO}(3),+} \simeq \mathcal{M}^{k=1}_{\mathrm{SU}(2)} \simeq \mathbb{C}\times \mathbb{C}P^1\,, \quad \mathcal{M}^{k=1}_{\mathrm{SO}(3),-} \simeq \mathbb{C}\,. \tag{3.118}$$

Note that f_3 always maps the moduli matrix of $G' = \mathrm{SU}(2)$ to that of $G' = \mathrm{SO}(3)$ with $Q_{\mathbb{Z}_2} = +1$.

We have seen very similar dual weight diagrams for $k=2$, $\mathrm{SO}(2), \mathrm{USp}(2)$ and $k=1$, $\mathrm{SO}(3)$ vortices. All of them consist of three sites on a straight line. However, when the connectedness is taken into account, they are quite different, see figure 11. The three points are isolated in the $\mathrm{SO}(2)$ case while they are all connected in the $\mathrm{USp}(2)$ case. In the case of $\mathrm{SO}(3)$, they split into two diagrams. One is a singlet and the other has two sites mutually connected, which describes $\mathbb{C}P^1$.

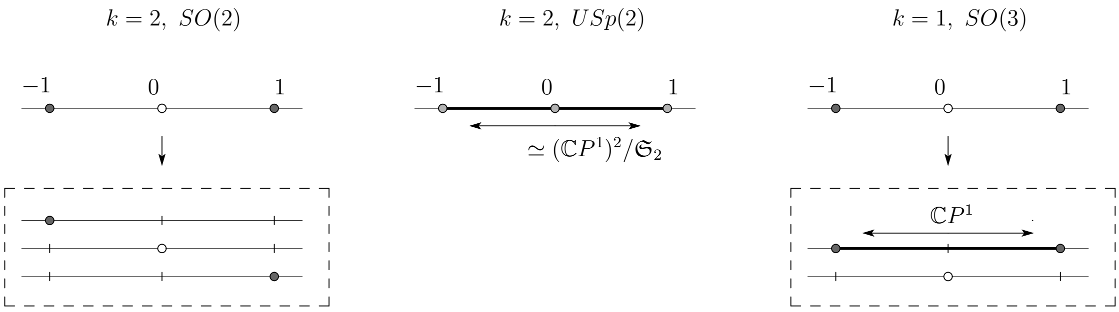

Figure 11. $k=1$ SO(3) and $k=2$ SO(2), USp(2).

$k=1$ local vortex for $G' = \mathrm{SO}(5)$

Finally, we move on to the second simplest case of odd SO vortices: $G' = \mathrm{SO}(5)$. Let us first list all the patches, starting with those having $Q_{\mathbb{Z}_2} = +1$:

$$H_0^{(0,0)} = z\mathbf{1}_5 + A\,, \tag{3.119}$$

$$H_0^{(1,1)} = \begin{pmatrix} z^2 & 0 & 0 & 0 & 0 \\ 0 & z^2 & 0 & 0 & 0 \\ -c_3^2 & -c_1 z + c_2 - c_3 c_4 & 1 & 0 & \sqrt{2}c_3 \\ c_1 z - c_2 - c_3 c_4 & -c_4^2 & 0 & 1 & \sqrt{2}c_4 \\ -\sqrt{2}c_3 z & -\sqrt{2}c_4 z & 0 & 0 & z \end{pmatrix}, \tag{3.120}$$

where

$$A = \begin{pmatrix} -a_1a_2 - a_3a_4 & -a_4^2 & 0 & a_1^2 & \sqrt{2}a_1a_4 \\ a_3^2 & -a_1a_2 + a_3a_4 & -a_1^2 & 0 & -\sqrt{2}a_1a_3 \\ 0 & a_2^2 & a_1a_2 + a_3a_4 & -a_3^2 & \sqrt{2}a_2a_3 \\ -a_2^2 & 0 & a_4^2 & a_1a_2 - a_3a_4 & \sqrt{2}a_2a_4 \\ -\sqrt{2}a_2a_3 & -\sqrt{2}a_2a_4 & -\sqrt{2}a_1a_4 & \sqrt{2}a_1a_3 & 0 \end{pmatrix}. \tag{3.121}$$

The patches $H_0^{(1,-1)}$, $H_0^{(-1,1)}$ and $H_0^{(-1,-1)}$ can be obtained from $H_0^{(1,1)}$ by the permutations (3.10). This means that the four patches $\{H_0^{(1,1)}, H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}\}$ are on an SO(5) orbit and they are certainly connected. By the general discussion in the previous section, we know that also $H^{(0,0)}$ and all the other four patches are connected. This can be seen explicitly by studying the transition functions among all these patches:

$$H_0^{(1,1)} = V^{(1,1),(0,0)} H_0^{(0,0)}, \tag{3.122}$$

$$V^{(1,1),(0,0)} = \begin{pmatrix} z + \frac{c_2 + c_3c_4}{c_1} & \frac{c_4^2}{c_1} & 0 & -\frac{1}{c_1} & -\frac{\sqrt{2}c_4}{c_1} \\ -\frac{c_3^2}{c_1} & z + \frac{c_2 - c_3c_4}{c_1} & \frac{1}{c_1} & 0 & \frac{\sqrt{2}c_3}{c_1} \\ 0 & -c_1 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 \\ -\sqrt{2}c_3 & -\sqrt{2}c_4 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{cases} a_1 = \pm\frac{1}{\sqrt{c_1}} \\ a_i = \pm\frac{c_i}{\sqrt{c_1}} \quad (i = 2,3,4)\,, \end{cases} \tag{3.123}$$

where the same sign has to be chosen for all the transition functions. This means that the moduli space for the minimal vortex with $Q_{\mathbb{Z}_2} = +1$ in $G' = \mathrm{SO}(5)$ is

$$\mathcal{M}^{k=1}_{\mathrm{SO}(5),+} = \mathbb{C} \times W\mathbb{C}P^4_{(2,1,1,1,1)} \simeq \mathbb{C} \times \mathbb{C}P^4/\mathbb{Z}_2 \,, \tag{3.124}$$

where the subscript $(2,1,1,1,1)$ denotes the $\mathrm{U}(1)^{\mathbb{C}}$ charges. The weighted complex projective space $W\mathbb{C}P^4_{(2,1,1,1,1)}$ is defined by the following equivalence relation among five complex parameters ϕ_i (i.e. the homogeneous coordinates)

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \sim (\lambda^2\phi_1, \lambda\phi_2, \lambda\phi_3, \lambda\phi_4, \lambda\phi_5)\,, \quad \lambda \in \mathbb{C}^* \,. \tag{3.125}$$

On the other hand, the patches corresponding to $Q_{\mathbb{Z}_2} = -1$ take the form

$$H_0^{(1,0)} = \begin{pmatrix} z^2 & 0 & 0 & 0 & 0 \\ -b_1 z & z & 0 & 0 & 0 \\ -b_1 b_2 - b_3^2 & b_2 & 1 & b_1 & \sqrt{2}b_3 \\ -b_2 z & 0 & 0 & z & 0 \\ -\sqrt{2}b_3 z & 0 & 0 & 0 & z \end{pmatrix} . \tag{3.126}$$

The remaining patches $H_0^{(-1,0)}$, $H_0^{(0,1)}$ and $H_0^{(0,-1)}$ are obtained by permutations (3.10) from $H_0^{(1,0)}$. Since all of them are on an SO(5) orbit, the moduli space of the $k = 1$ vortices with $Q_{\mathbb{Z}_2} = -1$ is

$$\mathcal{M}^{k=1}_{\mathrm{SO}(5),-} = \mathbb{C} \times \frac{\mathrm{SO}(5)}{\mathrm{U}(1) \times \mathrm{SO}(3)} \simeq \mathcal{M}^{k=1}_{\mathrm{USp}(4)} = \mathbb{C} \times \frac{\mathrm{USp}(4)}{\mathrm{U}(2)} \,. \tag{3.127}$$

The following V-transformation from the $(1,0)$-patch to the $(-1,0)$-patch is

$$H^{(-1,0)}(z) = V^{(-1,0),(1,0)}(z) H^{(1,0)}(z)\,, \tag{3.128}$$

$$V^{(-1,0),(1,0)} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\Xi & 0 & 0 \\ 0 & \frac{{c'}^2}{\Xi} & a'z & -\frac{2{a'}^2}{\Xi} & -\frac{2a'c'}{\Xi} \\ -\frac{2}{\Xi} & -\frac{2b'z}{\Xi} & z^2 & -\frac{2a'z}{\Xi} & -\frac{2c'z}{\Xi} \\ 0 & -\frac{2{b'}^2}{\Xi} & b'z & \frac{{c'}^2}{\Xi} & -\frac{2b'c'}{\Xi} \\ 0 & -\frac{2b'c'}{\Xi} & c'z & -\frac{2a'c'}{\Xi} & 1 - \frac{2{c'}^2}{\Xi} \end{pmatrix} , \tag{3.129}$$

$$\Xi \equiv 2a'b' + {c'}^2 \,. \tag{3.130}$$

The transition functions are as follows

$$a = -\frac{2a'}{\Xi}\,, \quad b = -\frac{2b'}{\Xi}\,, \quad c = \frac{2c'}{\Xi} \,. \tag{3.131}$$

4 Semi-local vortices

We now turn to the more general type of solutions, by relaxing the strong condition (2.64). Namely, we shall make use of only the weak condition (2.22) to define our vortices. As will be seen shortly, this leads to a larger class of solutions: the so-called semi-local vortices.

All our vortices including the semi-local ones being BPS saturated, can be analyzed by using the moduli matrix $H_0(z)$. The latter has the general properties:

- it is an $N_{\rm C} \times N_{\rm F}$ complex matrix;
- all of its elements are polynomials in z. The algorithm given in ref. [24] implies that it is sufficient to consider only polynomials as holomorphic functions;
- it is defined only up to the V-equivalence relation, eq. (2.26);
- it is subject to the weak condition, eq. (2.22).

The moduli parameters ϕ^i for a BPS vortex solution emerge as coefficients in $H_0(z)$ and thus the moduli space of the solutions is defined by the above properties only. Of course, all the matrices which we found in section 3 for the local vortices satisfy these conditions a fortiori. Specifically, one can easily check that the special point $H_0^{(\tilde{\mu}_1,\ldots,\tilde{\mu}_M)}$ in eq. (2.36) satisfies the weak condition.

In the strong coupling limit $e, g \to \infty$, the master equations (2.24) and (2.25) are exactly solved by $\Omega' = \Omega'_\infty, \omega = \omega_\infty$ in eq. (2.28) and the energy density and Kähler potential for the effective action for the vortices (lumps) are given by [22]

$$\mathcal{E} = 2\partial\bar{\partial}\mathcal{K}\,, \quad K(\phi^i, \phi^{i*}) = \int d^2x\;\mathcal{K}\,, \quad \mathcal{K} = \xi \log \operatorname{Tr}\left[\sqrt{I_{G'} I_{G'}^\dagger}\right]\,, \tag{4.1}$$

with the G'-invariant $I_{G'} = H_0^{\rm T}(z) J H_0(z)$. Even in the case of finite gauge couplings, these are considered to be good approximations when $m_{e,g}L \gg 1$ where L is the typical distance from the core of the vortices. By substituting a typical form of $H_0(z)$ into the above formula, one can obtain multiple peaks in the energy profile even for a minimal winding vortex ($k = 1$). We call these interesting multi-peak solutions *fractional vortices*. These will be discussed in a separate paper [53]. Before explicitly studying the semi-local vortices, let us first solve some technical problems left out from the previous section.

4.1 Dimension of the moduli space

The index theorem discussed in appendix A tells us that our moduli space has dimension:

$$\dim_{\mathbb{C}}\left(\mathcal{M}_{G',k}\right) = \frac{kN^2}{n_0} = \nu\, N^2\,. \tag{4.2}$$

This dimension should coincide with that of the space spanned by the moduli in $H_0(z)$, if the master equations have a unique solution for a given $H_0(z)$. It is easy to confirm this by considering the vicinity of a special point of the moduli space.

Let us find the general form of H_0 in the vicinity of the special point (2.36) by perturbing H_0. For definiteness, let us consider the perturbation around $H_0^{(\frac{k}{2},\ldots,\frac{k}{2})}$:

$$H_0^{(\frac{k}{2},\ldots,\frac{k}{2})} + \delta H_0 = \begin{pmatrix} z^k \mathbf{1}_M & \\ & \mathbf{1}_M \end{pmatrix} + \begin{pmatrix} \delta A(z) & \delta C(z) \\ \delta B(z) & \delta D(z) \end{pmatrix}\,, \tag{4.3}$$

where $\delta A(z), \delta B(z), \delta C(z)$ and $\delta D(z)$ are $M \times M$ matrices whose elements are holomorphic functions of z with small (infinitesimal) coefficients.[16] Not all of the fluctuations are

[16]Notice that here we are considering fluctuations around a k-vortex configuration with even parity. The generalization to the odd case is discussed at the end of the section.

independent though: we must fix them uniquely by using the V-equivalence (2.26). The infinitesimal V-transformation satisfies the condition $\delta V^{\mathrm{T}}(z)J + J\delta V(z) = 0$ which just represents the algebra of $\mathrm{SO}(2M,\mathbb{C}), \mathrm{USp}(2M,\mathbb{C})$ and can be expressed as

$$\delta V(z) = \begin{pmatrix} \delta L(z) & \delta N_{A,S}(z) \\ \delta M_{A,S}(z) & -\delta L^{\mathrm{T}}(z) \end{pmatrix} . \tag{4.4}$$

Again $\delta L(z), \delta M_{A,S}(z)$ and $\delta N_{A,S}(z)$ are $M \times M$ matrices whose elements are holomorphic in z and their coefficients are infinitesimally small. Acting with this infinitesimal V-transformation on the moduli matrix

$$\delta V(z) H_0^{(\frac{k}{2},\ldots,\frac{k}{2})} + \delta H_0 \simeq \begin{pmatrix} z^k \delta L(z) & \delta N_{A,S}(z) \\ z^k \delta M_{A,S}(z) & -\delta L^{\mathrm{T}}(z) \end{pmatrix} + \begin{pmatrix} \delta A(z) & \delta C(z) \\ \delta B(z) & \delta D(z) \end{pmatrix} , \tag{4.5}$$

we can set $\delta D(z) \to 0$, $\delta C \to \delta C_{S,A}(z)$ and $\delta B(z) \to \delta B_{S,A}(z) + \delta b(z)$ yielding:

$$\delta H_0 = \begin{pmatrix} \delta A(z) & \delta C_{S,A}(z) \\ \delta B_{S,A}(z) + \delta b(z) & 0 \end{pmatrix} . \tag{4.6}$$

Note that we have adopted the notation that $\delta X(z)$ stands for a general polynomial function while $\delta x(z)$ denotes a holomorphic function whose degree is less than the vortex number k. Now the V-transformation is completely fixed, and one can determine the true degrees of freedom of the fluctuations. The infinitesimal form of the weak condition (2.22) is

$$\delta H_0^{\mathrm{T}}(z) J H_0(z) + H_0(z) J \delta H_0(z) = \mathcal{O}(z^{k-1}) .$$

This leads to $\delta A \to \delta a(z)$, $\delta C_{S,A}(z) \to \delta c_{S,A}(z)$, $\delta B_{S,A}(z) \to 0$ and $\delta b(z) \to \delta b_{A,S}(z)$:

$$\delta H_0(z) = \begin{pmatrix} \delta a(z) & \delta c_{S,A}(z) \\ \delta b_{A,S}(z) & 0 \end{pmatrix} . \tag{4.7}$$

These are good coordinates in the vicinity of the special point

$$H_0^{(\frac{k}{2},\ldots,\frac{k}{2})} = \mathrm{diag}\left(z^k, \ldots, z^k, 1, \ldots, 1\right) .$$

Of course, this is only a local description but it is sufficient for counting the dimensions of the moduli space. The complex dimension is the number of the complex parameters in the fluctuations

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-semi-local}}_{\mathrm{SO}(2M),\mathrm{USp}(2M)} = 2kM^2 . \tag{4.8}$$

In order to restrict the solutions to the local vortices, one further imposes the following conditions:

$$\delta a(z) \to \delta P(z) \mathbf{1}_M , \quad \delta c_{S,A}(z) \to 0 , \tag{4.9}$$

with an arbitrary polynomial $\delta P(z)$ of order $(k-1)$. This leads to the dimension of the k local vortex moduli:

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-local}}_{\mathrm{SO}(2M),+} = k\left(1 + \frac{M(M-1)}{2}\right) , \tag{4.10}$$

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-local}}_{\mathrm{USp}(2M)} = k\left(1 + \frac{M(M+1)}{2}\right) . \tag{4.11}$$

In a similar way, one can count the dimension in the vicinity of the special point of positive chirality $(k,\ldots,k)$[17] for the SO$(2M+1)$ case and obtain

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-semi-local}}_{\mathrm{SO}(2M+1),+} = k\,(2M+1)^2 \ , \quad \dim_{\mathbb{C}} \mathcal{M}^{k\text{-local}}_{\mathrm{SO}(2M+1),+} = k\left(M^2+1\right) . \tag{4.12}$$

Notice that these results can be considered as a non-trivial consistency check for the moduli matrix formalism. In fact, by physical arguments, we always expect the following relation among the dimensions of the moduli spaces:

$$\dim_{\mathbb{C}} \mathcal{M}_k = k \dim_{\mathbb{C}} \mathcal{M}_{k=1}, \tag{4.13}$$

which is valid both for the local and semi-local case. This relations can be readily used to generalize the above equations to the other cases, including special points with odd chirality.

4.2 The $k = 1$ semi-local vortex in $G' = \mathrm{SO}(2M), \mathrm{USp}(2M)$ theories

Let us study the minimal-winding semi-local vortex in this section. The $k = 1$ vortex is special in the sense that all the fluctuations in eq. (4.7) can actually be promoted to finite parameters. Namely, the $H_0^{(\frac{1}{2},\ldots,\frac{1}{2})}$-patch is obtained by just replacing the small fluctuation $\delta a(z), \delta b_{A,S}(z), \delta c_{S,A}(z)$ by finite constant parameters $A, B_{A,S}, C_{S,A}$, respectively:

$$H_0^{(\frac{1}{2},\ldots,\frac{1}{2})}(z) = \begin{pmatrix} z\mathbf{1}_M + A & C_{S,A} \\ B_{A,S} & \mathbf{1}_M \end{pmatrix} . \tag{4.14}$$

One can verify that this indeed satisfies the weak condition (2.22) for $k = 1$. Notice that the above matrix can also be rewritten as

$$H_0^{(\frac{1}{2},\ldots,\frac{1}{2})}(z) = \tilde{U}_C \begin{pmatrix} z\mathbf{1}_M + \tilde{A} & \\ & \mathbf{1}_M \end{pmatrix} U_B \,, \tag{4.15}$$

where we have defined

$$\tilde{A} \equiv A - C_{S,A}B_{A,S} \,, \quad U_B \equiv \begin{pmatrix} \mathbf{1}_M & \\ B_{A,S} & \mathbf{1}_M \end{pmatrix} , \quad \tilde{U}_C \equiv \begin{pmatrix} \mathbf{1}_M & C_{S,A} \\ & \mathbf{1}_M \end{pmatrix} . \tag{4.16}$$

[17] Around other special points this strategy may not work in the local case. Other special points may sit on an intersection of two different submanifolds and one cannot make a distinction between the fluctuations among them. It is possible, in any case, to identify, case by case, a special point which does not lie on an intersection. However, one might sometimes need to include quadratic fluctuations, in order to implement correctly the strong condition.

When A is proportional to the unit matrix and $C_{S,A}$ is zero, that is, corresponding to a local vortex (3.8), U_B corresponds to the Nambu-Goldstone modes associated with the symmetry breaking $G'_{\rm C+F} \to {\rm U}(M)$. It is remarkable that this is not always the case for general semi-local configurations since a non-vanishing $\tilde{A}$ and $C_{S,A}$ break ${\rm U}(M)$ further down. In general, the symmetry breaking is $G'_{\rm C+F} \to \mathbb{Z}_{n_0}$.

Let us next consider the transition functions between two different patches. As we did for the local vortices in section 3, the other patches can be obtained as in eq. (3.9), i.e. via the permutation matrix P_r defined in eq. (3.10). Transition functions are always obtained by means of the V-transformations as in eq. (2.26)

$$H'_0(z) = V(z) H_0(z)\,, \quad V(z) \equiv V_e\, V'(z)\,, \quad V_e \in \mathbb{C}^*\,,\ V'(z) \in G'^{\mathbb{C}}\,. \tag{4.17}$$

For example, consider two patches, $H_0^{(\frac{1}{2},\ldots,\frac{1}{2})}(z)$ given by eq. (4.15) and

$$H_0^{(\overbrace{-\frac{1}{2},\ldots,-\frac{1}{2}}^{r},\overbrace{\frac{1}{2},\ldots,\frac{1}{2}}^{M-r})}(z) = P_r^{-1} H_0^{(\frac{1}{2},\ldots,\frac{1}{2})\prime}(z) P_r\,, \tag{4.18}$$

$$H_0^{(\frac{1}{2},\ldots,\frac{1}{2})\prime}(z) = \tilde{U}_{C'} \begin{pmatrix} z\mathbf{1}_M + \tilde{A}' & \\ & \mathbf{1}_M \end{pmatrix} U_{B'}\,. \tag{4.19}$$

The equation (4.17) in this case reads

$$\begin{pmatrix} z\mathbf{1}_M + \tilde{A}' & \\ & \mathbf{1}_M \end{pmatrix} U_{B'} P_r U_{-B} = \tilde{U}_{-C'} P_r V \tilde{U}_C \begin{pmatrix} z\mathbf{1}_M + \tilde{A} & \\ & \mathbf{1}_M \end{pmatrix}. \tag{4.20}$$

The transition functions will be determined by this condition together with

$$(U_{B'} P_r U_{-B})^{\rm T} J (U_{B'} P_r U_{-B}) = J\,, \quad \text{and} \quad (P_r V)^{\rm T} J (P_r V) = J\,.$$

The solution to these conditions are of the form

$$U_{B'} P_r U_{-B} = \begin{pmatrix} a & a\, d_{A,S} \\ 0 & (a^{-1})^{\rm T} \end{pmatrix}, \quad \tilde{U}_{-C'} P_r V \tilde{U}_C = \begin{pmatrix} a & (z\mathbf{1}_M + \tilde{A}')\, a\, d_{A,S} \\ 0 & (a^{-1})^{\rm T} \end{pmatrix}, \tag{4.21}$$

with $a \in {\rm GL}(M,\mathbb{C})$ and $d_{A,S}$ is an $M \times M$ (anti)symmetric matrix and

$$\tilde{A}' = a\, \tilde{A}\, a^{-1}\,, \tag{4.22}$$

$$C'_{S,A} = a \left[C_{S,A} - \frac{1}{2} \left(\tilde{A}\, d_{A,S} - d_{A,S}\, \tilde{A}^{\rm T} \right) \right] a^{\rm T}\,. \tag{4.23}$$

Notice that ${\rm Tr}\tilde{A}$ is invariant. The final step is to determine $a, d_{A,S}$ and the transition function for $B'_{A,S}$ by investigating the concrete form of U_B

$$U_B = \begin{pmatrix} \mathbf{1}_r & & & \\ & \mathbf{1}_{M-r} & & \\ b_1 & b_2 & \mathbf{1}_r & \\ -\epsilon\, b_2^{\rm T} & b_3 & & \mathbf{1}_{M-r} \end{pmatrix}, \quad b_{1,3}^{\rm T} = -\epsilon\, b_{1,3}\,, \tag{4.24}$$

and analogously for $U_{B'}$. Plugging this into the left hand side of the first equation in (4.21), one obtains the following result:

$$a = \begin{pmatrix} -\epsilon\, b_1 & -\epsilon\, b_2 \\ 0 & \mathbf{1}_{M-r} \end{pmatrix}, \quad d_{A,S} = \begin{pmatrix} -b_1^{-1} & \\ & \mathbf{0}_{M-r} \end{pmatrix}. \tag{4.25}$$

The transition functions between $B_{A,S}$ and $B'_{A,S}$ are indeed the same as those of the local vortex in eq. (3.13)

$$b'_1 = \epsilon\, b_1^{-1}, \quad b'_2 = b_1^{-1} b_2, \quad b'_3 = b_3 + \epsilon\, b_2^{\mathrm{T}} b_1^{-1} b_2 . \tag{4.26}$$

We again observe an important result from the first equation in (4.21). It tells us that

$$\det P_r = +1, \tag{4.27}$$

thus there exist two copies of the moduli space, which are disconnected even in the larger space including the semi-local vortices, in the case of $G' = \mathrm{SO}(2M)$. It is of course due to the $\mathbb{Z}_2$ parity (see section 2.3). As in the case of the local vortices in $G' = \mathrm{SO}(2M)$ theory discussed earlier, the patches with different $\mathbb{Z}_2$-parity are disconnected.

4.2.1 Example: $G' = \mathrm{SO}(4)$

Let us give an example in the $G' = \mathrm{SO}(4)$ theory. The patches with $\mathbb{Z}_2$-parity $+1$ are

$$H_0^{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} z+a & b & e & f \\ c & z+d & f & g \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}, \tag{4.28}$$

$$H_0^{(-\frac{1}{2},-\frac{1}{2})} = \begin{pmatrix} 1 & 0 & 0 & i' \\ 0 & 1 & -i' & 0 \\ e' & f' & z+a' & b' \\ f' & g' & c' & z+d' \end{pmatrix}. \tag{4.29}$$

These patches are connected by the V-transformation (2.26) $H_0^{(-\frac{1}{2},-\frac{1}{2})} = V^{(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2})} H_0^{(\frac{1}{2},\frac{1}{2})}$,

$$V^{(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} 0 & 0 & 0 & i' \\ 0 & 0 & -i' & 0 \\ 0 & \frac{1}{i'} & z+\frac{a'+d'}{2} & 0 \\ -\frac{1}{i'} & 0 & 0 & z+\frac{a'+d'}{2} \end{pmatrix}, \tag{4.30}$$

The explicit form of the transition function (the relation between the primed and unprimed parameters) is given in eq. (D.1).

There are two more patches for the vortex with $\mathbb{Z}_2$-parity -1 and are described by the moduli matrices

$$H_0^{(\frac{1}{2},-\frac{1}{2})} = \begin{pmatrix} z+a'' & f'' & e'' & b'' \\ -i'' & 1 & 0 & 0 \\ 0 & 0 & 1 & i'' \\ c'' & g'' & f'' & z+d'' \end{pmatrix} \tag{4.31}$$

$$H_0^{(-\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} 1 & i''' & 0 & 0 \\ f''' & z+d''' & b''' & e''' \\ g''' & c''' & z+a''' & f''' \\ 0 & 0 & -i''' & 1 \end{pmatrix} . \tag{4.32}$$

These two patches are connected in the same way as the two with positive chirality. In fact they define another copy of the same space. In agreement with the general results found above, neither one of the even patches: $H_0^{(-\frac{1}{2},-\frac{1}{2})}$, $H_0^{(\frac{1}{2},\frac{1}{2})}$, is connected with one of the odd, $H_0^{(\frac{1}{2},-\frac{1}{2})}$ and $H_0^{-(\frac{1}{2},\frac{1}{2})}$. One can easily see that there does not exist any V-transformation connecting them. One may construct a holomorphic matrix $X(z)$ which satisfies, for example, $H_0^{(\frac{1}{2},-\frac{1}{2})} = X(z)H_0^{(\frac{1}{2},\frac{1}{2})}$, however, violating the condition $X(z) \in \mathrm{SO}(4,\mathbb{C})$.

4.3 The $k=2$ semi-local vortices

Consider now the patches associated with the $k=2$ (doubly-wound) vortices. Let us begin with infinitesimal fluctuations around the special point

$$H_0^{(\overbrace{1,\ldots,1}^{r},\overbrace{0,\ldots,0}^{M-r})} = \begin{pmatrix} z^2\mathbf{1}_r & & & \\ & z\mathbf{1}_{M-r} & & \\ & & \mathbf{1}_r & \\ & & & z\mathbf{1}_{M-r} \end{pmatrix} \rightarrow H_0^{(1,\ldots,1,0,\ldots,0)} + \delta H_0(z) . \tag{4.33}$$

In order to get rid of the unphysical degrees of freedom in the fluctuations δH_0, let us consider an infinitesimal V-transformation (2.26)

$$\delta V = \begin{pmatrix} \delta K_{11} & \delta M_{11} & \delta K_{12;A,S} & \delta M_{12} \\ \delta L_{11} & \delta N_{11} & -\epsilon\,\delta M_{12}^{\mathrm{T}} & \delta N_{12;A,S} \\ \delta K_{21;A,S} & \delta M_{21} & -\delta K_{11}^{\mathrm{T}} & -\delta L_{11}^{\mathrm{T}} \\ -\epsilon\,\delta M_{21}^{\mathrm{T}} & \delta N_{21;A,S} & -\delta M_{11}^{\mathrm{T}} & -\delta N_{11}^{\mathrm{T}} \end{pmatrix} . \tag{4.34}$$

Acting with the V-transformation on the perturbed moduli matrix, we find

$$\delta H_0 \sim \delta H_0 + \delta V H_0^{(1,\ldots,1,0,\ldots,0)} . \tag{4.35}$$

Since the explicit form of $\delta V H_0^{(1,\ldots,1,0,\ldots,0)}$ is

$$\delta V H_0^{(1,\ldots,1,0,\ldots,0)} = \begin{pmatrix} z^2\delta K_{11} & z\delta M_{11} & \delta K_{12;A,S} & z\delta M_{12} \\ z^2\delta L_{11} & z\delta N_{11} & -\epsilon\,\delta M_{12}^{\mathrm{T}} & z\delta N_{12;A,S} \\ z^2\delta K_{21;A,S} & z\delta M_{21} & -\delta K_{11}^{\mathrm{T}} & -z\delta L_{11}^{\mathrm{T}} \\ -z^2\epsilon\,\delta M_{21}^{\mathrm{T}} & z\delta N_{21;A,S} & -\delta M_{11}^{\mathrm{T}} & -z\delta N_{11}^{\mathrm{T}} \end{pmatrix} ,$$

the physical degrees of freedom in the fluctuations can be expressed as

$$\delta H_0 = \begin{pmatrix} \delta A_{11} & \delta C_{11} & \delta A_{12;S,A} & \delta C_{12} \\ \delta B_{11} & \delta D_{11} & 0 & \delta D_{12;S,A} + \delta d_{12;A,S} \\ \delta A_{21;S,A} + \delta a^{(1)}_{21;A,S} z + \delta a^{(0)}_{21;A,S} & \delta c_{21} & 0 & \delta c_{22} \\ \delta B_{21} & \delta D_{21;S,A} + \delta d_{21;A,S} & 0 & \delta d_{22} \end{pmatrix} ,$$

where δX denotes a generic holomorphic polynomial and δx stands for a constant matrix. The infinitesimal version of the weak condition (2.22)

$$\delta H_0^{\mathrm{T}}(z) J H_0(z) + H_0(z) J \delta H_0(z) = \mathcal{O}(z)\,, \tag{4.36}$$

turns out to be equivalent to the following conditions

$$\begin{gathered}\{\delta D_{11},\ \delta D_{21;S,A},\ \delta D_{12;S,A}\} = \mathcal{O}(1)\,,\\ \{\delta A_{11},\ \delta C_{11},\ \delta A_{12;S,A},\ \delta C_{12}\} = \mathcal{O}(z)\,,\\ \delta A_{21;S,A} = 0\,, \quad \delta B_{11} = -\delta c_{22}^{\mathrm{T}} z + \delta b_{11}\,, \quad \delta B_{21} = -\epsilon\, \delta c_{21}^{\mathrm{T}} z + \delta b_{21}\,.\end{gathered} \tag{4.37}$$

We thus find the generic form of the fluctuations in the vicinity of the special point $H_0^{(1,\ldots,1,0\cdots,0)}$ as

$$\delta H_0 = \begin{pmatrix} \delta a_{11}^{(1)} z + \delta a_{11}^{(0)} & \delta c_{11}^{(1)} z + \delta c_{11}^{(0)} & \delta a_{12;S,A}^{(1)} z + \delta a_{12;S,A}^{(0)} & \delta c_{12}^{(1)} z + \delta c_{12}^{(0)} \\ -\delta c_{22}^{\mathrm{T}} z + \delta b_{11} & \delta d_{11} & 0 & \delta d_{12} \\ \delta a_{21;A,S}^{(1)} z + \delta a_{21;A,S}^{(0)} & \delta c_{21} & 0 & \delta c_{22} \\ -\epsilon\, \delta c_{21}^{\mathrm{T}} z + \delta b_{21} & \delta d_{21} & 0 & \delta d_{22} \end{pmatrix}. \tag{4.38}$$

Let us count the dimensions of the moduli space. We have six matrices $\delta a_{ij}^{(\alpha)}$ of size $r \times r$, two matrices δb_{ij} of size $(M-r) \times r$, six matrices $\delta c_{ij}^{(\alpha)}$ of size $r \times (M-r)$ and four matrices δd_{ij} of the size $(M-r) \times (M-r)$. Thus summing up we obtain the correct dimension

$$\dim_{\mathbb{C}} \left[\mathcal{M}^{\text{2-semi-local}}_{\mathrm{SO}(2M),\mathrm{USp}(2M)}\right] = 4M^2\,. \tag{4.39}$$

The next task is to find the coordinate patches with *finite* parameters (i.e. large fluctuations). To this end, let us naively promote all the small fluctuations in eq. (4.38) to finite parameters as $\delta x \to x$ (as was done in the case of the minimal semi-local vortices):

$$H_0 = \begin{pmatrix} z^2 \mathbf{1}_r + a_{11}^{(1)} z + a_{11}^{(0)} & c_{11}^{(1)} z + c_{11}^{(0)} & a_{12;S,A}^{(1)} z + a_{12;S,A}^{(0)} & c_{12}^{(1)} z + c_{12}^{(0)} \\ -c_{22}^{\mathrm{T}} z + b_{11} & z \mathbf{1}_{M-r} + d_{11} & 0 & d_{12} \\ a_{21;A,S}^{(1)} z + a_{21;A,S}^{(0)} & c_{21} & \mathbf{1}_r & c_{22} \\ -\epsilon\, c_{21}^{\mathrm{T}} z + b_{21} & d_{21} & 0 & z \mathbf{1}_{M-r} + d_{22} \end{pmatrix}. \tag{4.40}$$

But such a procedure is inconsistent with the weak condition (2.22). Although $H_0^{\mathrm{T}} J H_0\big|_{\mathcal{O}(z^n)} = 0$ for $n \geq 3$, the terms of order $\mathcal{O}(z^2)$ turn out to be (z^2 times)

$$H_0^{\mathrm{T}} J H_0\big|_{\mathcal{O}(z^2)} = \begin{pmatrix} -2\Lambda_{S,A} & -a_{21;A,S}^{(1)} c_{11}^{(1)} & \mathbf{1}_r - a_{21;A,S}^{(1)} a_{12;S,A}^{(1)} & -a_{21;A,S}^{(1)} c_{12}^{(1)} \\ c_{11}^{(1)\mathrm{T}} a_{21;A,S}^{(1)} & 0 & 0 & \mathbf{1}_{M-r} \\ \epsilon\,(\mathbf{1}_r + a_{12;S,A}^{(1)} a_{21;A,S}^{(1)}) & 0 & 0 & 0 \\ c_{12}^{(1)\mathrm{T}} a_{21;A,S}^{(1)} & \epsilon\, \mathbf{1}_{M-r} & 0 & 0 \end{pmatrix}, \tag{4.41}$$

with

$$-2\Lambda_{S,A} \equiv a_{11}^{(1)\mathrm{T}} a_{21;A,S}^{(1)} - a_{21;A,S}^{(1)} a_{11}^{(1)} + c_{21} c_{22}^{\mathrm{T}} + \epsilon\, c_{22} c_{21}^{\mathrm{T}}\,. \tag{4.42}$$

This must be $H_0^{\mathrm{T}} J H_0\big|_{\mathcal{O}(z^2)} = J$, i.e. we have to eliminate the undesired terms, such that eq. (4.41) becomes exactly equal to J. To compensate the surplus terms, we add the following extra term

$$H_0^{\text{extra}} = \begin{pmatrix} \mathbf{0}_r & & & \\ & \mathbf{0}_{M-r} & & \\ \Lambda_{S,A} & a^{(1)}_{21;A,S} c^{(1)}_{11} & a^{(1)}_{21;A,S} a^{(1)}_{12;S,A} & a^{(1)}_{21;A,S} c^{(1)}_{12} \\ & & & \mathbf{0}_{M-r} \end{pmatrix} . \tag{4.43}$$

Finally we obtain the finite coordinate patch

$$\begin{aligned} &H_0^{(\overbrace{1,\dots,1}^{r},\overbrace{0,\dots,0}^{M-r})}(z) = \\ &\begin{pmatrix} z^2\mathbf{1}_r + a^{(1)}_{11} z + a^{(0)}_{11} & c^{(1)}_{11} z + c^{(0)}_{11} & a^{(1)}_{12;S,A} z + a^{(0)}_{12;S,A} & c^{(1)}_{12} z + c^{(0)}_{12} \\ -c^{\mathrm{T}}_{22} z + b_{11} & z\mathbf{1}_{M-r} + d_{11} & 0 & d_{12} \\ a^{(1)}_{21;A,S} z + a^{(0)}_{21;A,S} + \Lambda_{S,A} & c_{21} + a^{(1)}_{21;A,S} c^{(1)}_{11} & \mathbf{1}_r + a^{(1)}_{21;A,S} a^{(1)}_{12;S,A} & c_{22} + a^{(1)}_{21;A,S} c^{(1)}_{12} \\ -\epsilon\, c^{\mathrm{T}}_{21} z + b_{21} & d_{21} & 0 & z\mathbf{1}_{M-r} + d_{22} \end{pmatrix} . \end{aligned} \tag{4.44}$$

All other patches can be obtained by making use of the permutation (3.10):

$$H_0^{(\overbrace{1,\dots,1}^{r},\overbrace{0,\dots,0}^{M-r})}(z) \to P_{r'}^{-1} H_0^{(\overbrace{1,\dots,1}^{r},\overbrace{0,\dots,0}^{M-r})'}(z) P_{r'} \, . \tag{4.45}$$

Since the transition functions between the different patches of the $k=2$ semi-local vortices are rather complicated, we shall not discuss them in this paper; we limit ourselves to showing just a few simple examples below.

4.3.1 $G' = \mathrm{SO}(4)$

As in the case of the $k=2$ local vortices discussed in section 3.2.2, at least nine patches are needed to describe the $k=2$ semi-local vortices. They are divided into two disconnected parts as $9 = 5+4$ according to the $\mathbb{Z}_2$-parity. The five matrices corresponding to $Q_{\mathbb{Z}_2} = +1$ are $\{H_0^{(1,1)}, H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}, H_0^{(0,0)}\}$ and the four matrices with $Q_{\mathbb{Z}_2} = -1$ are $\{H_0^{(1,0)}, H_0^{(-1,0)}, H_0^{(0,1)}, H_0^{(0,-1)}\}$.

Let us start with the patches having $Q_{\mathbb{Z}_2} = +1$,

$$H_0^{(0,0)} = (z - z_0)\mathbf{1}_4 + D \, , \tag{4.46}$$

$$H_0^{(1,1)} = \begin{pmatrix} z^2\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix} + \begin{pmatrix} A_1 z + A_0 & C_{1S} z + C_{0S} \\ H_{1A} z + H_{0A} + \frac{1}{2}\left(H_{1A}A_1 - A_1^{\mathrm{T}} H_{1A}\right) & H_{1A} C_{1S} \end{pmatrix} , \tag{4.47}$$

where D is an arbitrary 4×4 matrix. The other patches $\{H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}\}$ can be obtained by the permutations (3.10) of $H_0^{(1,1)}$.

Now we can clearly see the difference between the local and semi-local vortices. Let us consider the $(0,0)$-patch. The patches for the local vortices are given in eq. (3.71) and those

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for the semi-local vortices in eq. (4.46). To avoid confusion, let us denote them by $(0,0)_{l+}$ and $(0,0)_{l-}$ for the former and $(0,0)_{sl}$ for the latter. Clearly, the $(0,0)_{l+}$ and $(0,0)_{l-}$ patches are unified into the $(0,0)_{sl}$-patch when the strong condition is relaxed to the weak one.

As explained in section 3.2.2, the $(0,0)_{l+}$ patch (with the $(1,1)$ and $(-1,-1)$ patches) and the $(0,0)_{l-}$-patch (with the $(-1,1)$ and $(1,-1)$ patches) correspond to two possible choices of the $\mathbb{Z}_2$-parities of the component vortices $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (\pm 1, \pm 1)$. This reflects the fact that any product of the moduli matrices for local vortices generates automatically local vortices. It is tempting to interpret the fact that the two spaces are disconnected as meaning that the $\mathbb{Z}_2$-parity of each component vortex is conserved. However, this is not the case for the semi-local vortices. Products of moduli matrices satisfying the weak condition (2.22) do not, in general, satisfy it. The $\mathbb{Z}_2$-parity of each vortex is therefore not conserved in the semi-local case.

Let us examine the transition functions between the $(1,1)$ and $(0,0)$-patches, explicitly. Notice, that we have already observed the connectedness between them, as it was indeed present in the case of the local vortices. Our aim to express the following complicated results is completeness of the calculations. Let us write down the moduli matrices as

$$H_0^{(1,1)} = \begin{pmatrix} z^2 + a_1' z + a_0' & b_1' z + b_0' & e_1' z + e_0' & f_1' z + f_0' \\ c_1' z + c_0' & z^2 + d_1' z + d_0' & f_1' z + f_0' & g_1' z + g_0' \\ c_1' i_1' & i_1' z + i_0' - \frac{1}{2} a_1' i_1' + \frac{1}{2} d_1' i_1' & 1 + f_1' i_1' & g_1' i_1' \\ -i_1' z - i_0' - \frac{1}{2} a_1' i_1' + \frac{1}{2} d_1' i_1' & -b_1' i_1' & -e_1' i_1' & 1 - f_1' i_1' \end{pmatrix}, \tag{4.48}$$

$$H_0^{(0,0)} = \begin{pmatrix} z + a_0 & b_0 & c_0 & d_0 \\ e_0 & z + f_0 & g_0 & h_0 \\ i_0 & j_0 & z + k_0 & l_0 \\ m_0 & n_0 & o_0 & z + p_0 \end{pmatrix}. \tag{4.49}$$

The transition functions are determined through a V-transformation (2.26) satisfying the relation $V^{(1,1),(0,0)} H_0^{(0,0)} = H_0^{(1,1)}$:

$$V^{(1,1),(0,0)} = \begin{pmatrix} z + \frac{1}{2} a_1' + \frac{1}{2} d_1' - \frac{i_0'}{i_1'} & 0 & 0 & \frac{1}{i_1'} \\ 0 & z + \frac{1}{2} a_1' + \frac{1}{2} d_1' - \frac{i_0'}{i_1'} & -\frac{1}{i_1'} & 0 \\ 0 & i_1' & 0 & 0 \\ -i_1' & 0 & 0 & 0 \end{pmatrix}. \tag{4.50}$$

The transition functions connecting the patches $H_0^{(0,0)}$ and $H_0^{(1,1)}$ are thus given explicitly, see eq. (D.2).

The transition functions between the $(1,-1)$ and $(0,0)$-patches can be obtained by the permutation of the above $(1,1)$-$(0,0)$ system as

$$P^{-1} H_0^{(1,1)} P = H_0^{(1,-1)}, \quad P^{-1} H_0^{(0,0)} P = \tilde{H}_0^{(0,0)}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.51}$$

Therefore, the transition functions are easily found as

$$V^{(1,-1),(0,0)}\tilde{H}_0^{(0,0)} = H_0^{(1,-1)}\,, \quad V^{(1,-1),(0,0)} \equiv P^{-1}V^{(1,1),(0,0)}P\,. \tag{4.52}$$

The transition functions between the $(1,1)$ and $(1,-1)$-patches can be obtained by combining two transition functions given above.

Let us next show the transition functions between the patches with $\mathbb{Z}_2$-parity -1. The explicit form of the moduli matrix is given by

$$H_0^{(1,0)} = \begin{pmatrix} z^2 & & & \\ & z & & \\ & & 1 & \\ & & & z \end{pmatrix} + \begin{pmatrix} a_1z+a_0 & b_1z+b_0 & c_1z+c_0 & d_1z+d_0 \\ -e_1z+e_0 & f_0 & 0 & g_0 \\ -e_1i_1 & i_1 & 0 & e_1 \\ -i_1z+i_0 & j_1 & 0 & k_0 \end{pmatrix}\,. \tag{4.53}$$

The $(-1,0)$-patch can be obtained by acting with the permutation matrix on the $(1,1)$-patch as follows

$$H_0^{(-1,0)} = P^{-1}H_0^{(1,0)\prime}P\,, \qquad P = \begin{pmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix}\,. \tag{4.54}$$

The transition functions between these two patches are obtained by

$$V^{(-1,0),(1,0)}H_0^{(1,0)} = H_0^{(-1,0)}\,, \tag{4.55}$$

$$V^{(-1,0),(1,0)} = \begin{pmatrix} 0 & 0 & -i_1'e_1' & 0 \\ 0 & 0 & -e_1'z+e_0' & -\frac{e_1'}{i_1'} \\ -\frac{1}{e_1'i_1'} & \frac{1}{e_1'}\left(z-\frac{e_0'}{e_1'}\right) & \left(z-\frac{e_0'}{e_1'}\right)\left(z-\frac{i_0'}{i_1'}\right) & \frac{1}{i_1'}\left(z-\frac{i_0'}{i_1'}\right) \\ 0 & -\frac{i_1'}{e_1'} & -i_1'z+i_0' & 0 \end{pmatrix}\,. \tag{4.56}$$

The other transition functions between all the other patches are obtained through suitable permutations.

It can be shown that the patches with $Q_{\mathbb{Z}_2} = +1$ and those with $Q_{\mathbb{Z}_2} = -1$ are indeed disconnected. Let us take the example of the two moduli matrices $H_0^{(0,0)}$ and $H_0^{(1,0)}$. Assume that there exists a V-function such that

$$VH_0^{(0,0)} = H_0^{(1,0)}\,. \tag{4.57}$$

First we observe that V is a matrix whose elements are all at most of order z. This is due to $H_0^{(0,0)}$ having the term, $z\mathbf{1}_4$ and the highest power of $VH_0^{(0,0)}$ should not exceed 2 which is the highest degree of $H_0^{(1,0)}$. We can thus determine the linear term in z of V

$$V = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} z + \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}\,. \tag{4.58}$$

Furthermore, let us focus on the linear terms of z in eq. (4.57), i.e.,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} D + \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ -e_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i_1 & 0 & 0 & 1 \end{pmatrix} . \tag{4.59}$$

By comparison of the third row of both sides, we conclude that $(v_{31}, v_{32}, v_{33}, v_{34}) = (0, 0, 0, 0)$. However, $\det V = 0$ does not satisfy the requirement $V \in \mathrm{SO}(4, \mathbb{C})$: hence these two patches are disconnected.

4.4 The $k = 1$ semi-local vortex for $G' = \mathrm{SO}(2M + 1)$

The result of the index theorem (see appendix A) yields that the real dimension is $2k(2M+1)^2$ for the moduli space in $\mathrm{SO}(2M+1)$. Following the technology explained in section 4.3, it is straightforward to extend the results to the case of $G' = \mathrm{SO}(2M + 1)$. The moduli matrix for $k = 1$ in the $(\overbrace{1, \ldots, 1}^{r}, \overbrace{0, \ldots, 0}^{M-r})$-patch is the most general semi-local moduli matrix and is given by

$$H_0^{(\overbrace{1,\ldots,1}^{r},\overbrace{0,\ldots,0}^{M-r})}(z) =$$
$$\begin{pmatrix} z^2\mathbf{1}_r + a^{(1)}_{11} z + a^{(0)}_{11} & c^{(1)}_{11} z + c^{(0)}_{11} & a^{(1)}_{12;S} z + a^{(0)}_{12;S} & c^{(1)}_{12} z + c^{(0)}_{12} & e^{(1)}_{15} z + e^{(0)}_{15} \\ -c^{\mathrm{T}}_{22} z + b_{11} & z\mathbf{1}_{M-r} + d_{11} & 0 & d_{12} & e_{25} \\ a^{(1)}_{21;A} z + a^{(0)}_{21;A} + \Lambda_S & c_{21} + a^{(1)}_{21;A} c^{(1)}_{11} & \mathbf{1}_r + a^{(1)}_{21;A} a^{(1)}_{12;S} & c_{22} + a^{(1)}_{21;A} c^{(1)}_{12} & e_{35} + a^{(1)}_{21;A} e^{(1)}_{15} \\ -c^{\mathrm{T}}_{21} z + b_{21} & d_{21} & 0 & z\mathbf{1}_{M-r} + d_{22} & e_{45} \\ -e^{\mathrm{T}}_{35} z + e^{\mathrm{T}}_{31} & e^{\mathrm{T}}_{32} & 0 & e^{\mathrm{T}}_{34} & z + e_{55} \end{pmatrix} , \tag{4.60}$$

where we have defined

$$-2\Lambda_S \equiv a^{(1)\mathrm{T}}_{11} a^{(1)}_{21;A} - a^{(1)}_{21;A} a^{(1)}_{11} + c_{21} c^{\mathrm{T}}_{22} + c_{22} c^{\mathrm{T}}_{21} + e_{35} e^{\mathrm{T}}_{35} . \tag{4.61}$$

4.4.1 $G' = \mathrm{SO}(3)$

For $G' = \mathrm{SO}(3), k = 1$ there are 3 patches, viz. (1), (-1), (0). The moduli matrix for the (0)-patch is simply

$$H_0^{(0)} = z\mathbf{1}_3 + A\,, \tag{4.62}$$

where it is noteworthy to remark that the color+flavor symmetry is unbroken.

The moduli matrix for the (1)-patch is

$$H_0^{(1)} = \begin{pmatrix} z^2 + z_1 z + z_2 & a + fz & c + bz \\ -\frac{d^2}{2} & 1 & -d \\ e + dz & 0 & z - z_3 \end{pmatrix} , \tag{4.63}$$

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while the moduli matrix for the (-1)-patch is simply obtained by the permutation

$$H_0^{(-1)} = P H_0^{(1)} P^{-1}\,, \qquad \text{with } P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{4.64}$$

The patches (-1) and (1) are connected by a V-transformation given by

$$H'^{(1)}_0 = V^{(1),(-1)} H_0^{(-1)} \quad \text{with} \quad V^{(1),(-1)} = \begin{pmatrix} \frac{(e'+d'z')^2}{d'^2} & -\frac{2}{d'^2} & -\frac{2(e'+d'z')}{d'^2} \\ -\frac{d'^2}{2} & 0 & 0 \\ e'+d'z' & 0 & -1 \end{pmatrix}\,, \tag{4.65}$$

and the transition functions can be found in the appendix. The mass center of the system can be identified by taking the coefficient of the z^2 term of $\det H_0$. It is given by: $C.M. = -z_1' + z_3' + b'd' + d'^2 f'/2 = -z_1 + z_3 + bd + d^2 f/2$, which has a form that is invariant under the change of patch.

The patches (1) and (0) are disconnected. This can be seen from identifying the linear order of V

$$H_0^{(1)} = V H_0'^{(0)} = V\left(z\mathbf{1}_3 + A'\right) \qquad \Rightarrow \qquad V = z\ \mathrm{diag}(1,0,0) + V_{\mathrm{const}}\,. \tag{4.66}$$

Looking now at the linear order in z of the equation

$$\begin{pmatrix} z_1 & f & b \\ 0 & 0 & 0 \\ d & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A' + \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix}\,, \tag{4.67}$$

which reveals that the second row of V has to be zero, which takes V out of $\mathrm{SO}(3,\mathbb{C})$ and the patches are thus disconnected.

4.4.2 $G' = \mathrm{SO}(5)$

For SO(5) we have nine patches. The five having $\mathbb{Z}_2$ charge $+1$ are all connected and are described by the following moduli matrices

$$H^{(0,0)}(z) = z\mathbf{1}_5 + \begin{pmatrix} a_1' & a_2' & a_3' & a_4' & a_5' \\ b_1' & b_2' & b_3' & b_4' & b_5' \\ c_1' & c_2' & c_3' & c_4' & c_5' \\ d_1' & d_2' & d_3' & d_4' & d_5' \\ e_1' & e_2' & e_3' & e_4' & e_5' \end{pmatrix}\,, \tag{4.68}$$

$$H^{(1,1)}(z) = \begin{pmatrix} z^2+a_1z+b_1 & a_2z+b_2 & c_1z+d_1 & c_2z+d_2 & g_1z+h_1 \\ a_3z+b_3 & z^2+a_4z+b_4 & c_2z+d_2 & c_3z+d_3 & g_2z+h_1 \\ ea_3-\frac{i_1^2}{2} & ez+f-\frac{e(a_1-a_4)}{2}-\frac{i_1i_2}{2} & 1+ec_2 & ec_3 & i_1+eg_2 \\ -ez-f-\frac{e(a_1-a_4)}{2}-\frac{i_1i_2}{2} & -ea_2-\frac{i_2^2}{2} & -ec_1 & 1-ec_2 & i_2-eg_1 \\ -i_1z+j_1 & -i_2z+j_2 & 0 & 0 & z+y \end{pmatrix}\,, \tag{4.69}$$

with the rest being permutations of the latter. The moduli matrix $(0,0)$-patch is connected to the $(1,1)$-patch by the following V-transformation

$$H^{(1,1)}(z) = V^{(1,1),(0,0)}(z) H^{(0,0)}(z)\,, \tag{4.70}$$

$$V^{(1,1),(0,0)} = \begin{pmatrix} z+\frac{a_1+a_4}{2}-\frac{f}{e}-\frac{i_1 i_2}{2e} & -\frac{i_2^2}{2e} & 0 & \frac{1}{e} & \frac{i_2}{e} \\ \frac{i_1^2}{2e} & z+\frac{a_1+a_4}{2}-\frac{f}{e}+\frac{i_1 i_2}{2e} & -\frac{1}{e} & 0 & -\frac{i_1}{e} \\ 0 & e & 0 & 0 & 0 \\ -e & 0 & 0 & 0 & 0 \\ -i_1 & -i_2 & 0 & 0 & 1 \end{pmatrix}\,, \tag{4.71}$$

where the transition functions can be found in appendix D. There are four patches having $\mathbb{Z}_2$-charge -1, which are all connected. They are described by (and permutations of) the following moduli matrix

$$H^{(1,0)}(z) = \begin{pmatrix} z^2+a_1 z+a_2 & c_1 z+c_0 & b_1 z+b_0 & d_1 z+d_0 & i_1 z+i_0 \\ f_0-e_1 z & z+g_0 & 0 & g_1 & j_0 \\ -e_0 e_1-\frac{j_1^2}{2} & e_0 & 1 & e_1 & j_1 \\ f_1-e_0 z & g_2 & 0 & z+g_3 & j_2 \\ h_0-j_1 z & h_1 & 0 & h_2 & z+k \end{pmatrix}\,. \tag{4.72}$$

This patch is connected to $H^{(-1,0)}$ by the following V-transformation

$$H^{(-1,0)}(z) = V^{(-1,0),(1,0)}(z) H^{(1,0)}(z)\,, \tag{4.73}$$

$$V^{(-1,0),(1,0)} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\Xi & 0 & 0 \\ 0 & \frac{{j'}_1^2}{\Xi} & f'_0-e'_1 z & -\frac{2{e'}_1^2}{\Xi} & \frac{2e'_1 j'_1}{\Xi} \\ -\frac{2}{\Xi} & \frac{L_1(z)}{\Xi^2} & \frac{L_2(z)}{\Xi} & \frac{L_3(z)}{\Xi^2} & \frac{L_4(z)}{\Xi^2} \\ 0 & -\frac{2{e'}_0^2}{\Xi} & f'_1-e'_0 z & \frac{{j'}_1^2}{\Xi} & \frac{2e'_0 j'_1}{\Xi} \\ 0 & \frac{2e'_0 j'_1}{\Xi} & -h'_0+j'_1 z & \frac{2e'_1 j'_1}{\Xi} & 1-\frac{2{j'}_1^2}{\Xi} \end{pmatrix}\,, \tag{4.74}$$

$$\Xi \equiv 2e'_0 e'_1 + {j'}_1^2\,, \tag{4.75}$$

$$\frac{1}{2}L_1(z) \equiv f'_1 {j'}_1^2 - 2{e'}_0^2\left(f'_0 - e'_1 z\right) + e'_0 j'_1\left(j'_1 z - 2h'_0\right)\,, \tag{4.76}$$

$$L_2(z) \equiv {h'}_0^2 - 2h'_0 j'_1 z + 2f'_0\left(f'_1 - e'_0 z\right) + z\left(2e'_0 e'_1 z + {j'}_1^2 z - 2e'_1 f'_1\right)\,, \tag{4.77}$$

$$\frac{1}{2}L_3(z) \equiv f'_0 {j'}_1^2 - 2{e'}_1^2\left(f'_1 - e'_0 z\right) + e'_1 j'_1\left(-2h'_0 + j'_1 z\right)\,, \tag{4.78}$$

$$\frac{1}{2}L_4(z) \equiv j'_1\left(2e'_1 f'_1 + j'_1\left(h'_0 - j'_1 z\right)\right) - 2e'_0\left(e'_1\left(h'_0 + j'_1 z\right) - f'_0 j'_1\right)\,. \tag{4.79}$$

The patches of different chiralities are indeed disconnected, as we expected from topological reasons.

5 Conclusion and discussion

In this paper we have analyzed the BPS vortices appearing in $\mathrm{SO}(N) \times \mathrm{U}(1)$ and $\mathrm{USp}(2N) \times \mathrm{U}(1)$ gauge theories. The concrete model which our analysis is based upon can be regarded as the bosonic sector of the corresponding $\mathcal{N}=2$ gauge theories, but many of our conclusions are valid on much more general grounds. A short introduction to the construction of BPS vortices in a general gauge group has already been given by some of us [23].

It has been found that, in contrast to the vortices in $[\mathrm{SU}(N)\times\mathrm{U}(1)]/\mathbb{Z}_N \simeq \mathrm{U}(N)$ models studied extensively during the last several years, the vortex moduli in these theories contain certain other moduli, generally known as semi-local vortices, whose profile functions are characterized by their asymptotic, power-like behavior, whereas the standard ANO vortices (including their non-Abelian counterparts found in $\mathrm{U}(N)$ theories) have sharp, exponential cutoff to their transverse size. This is so even with the minimal number of matter fields, sufficient for the system to have a "color-flavor-locked" Higgs phase. The difference with the unitary gauge group case reflects the fact that, for a given dimension, the number of gauge degrees of freedom is less here, due to the fact that e.g., $\mathrm{SO}(2N), \mathrm{USp}(2N)$ groups constitute a strict subgroup $\mathrm{SU}(2N)$.

The existence of these semi-local extensions of the vortex moduli is related to the existence of non-trivial vacuum moduli of the system, and consequently, to the sigma model lumps which emerge in the strong gauge coupling limit of our vortices [22, 54]. In this limit a vortex solution collapses to a vacuum configuration everywhere on the transverse plane. It defines a map of a 2-cycle onto the moduli space of vacua, and is thus characterized by non-trivial elements of $\pi_2(M_{\mathrm{vac}})$. The existence of these semi-local moduli provides the vortex, even at finite coupling, with a very rich structure. In this paper we tried to uncover their general properties, with the help of concrete examples for the case of a few lower-rank groups. An interesting phenomenon concerns the emergence of fractional vortices, where a certain multi-peaked vortex configuration appear, even if the vortex, as a whole, has the topologically minimal winding allowed by the stability. These features will be discussed more extensively in a separate article [53].

Related to semi-local vortices is the issue of the non-normalizability of some of the moduli space parameters. In the case of $\mathrm{U}(N)$ vortices this question was solved completely [18], by using the general formula for the effective action of vortices in terms of the moduli matrix [29]. A part of this question was solved for a single vortex in SO and USp gauge theories in the lump limit [22]. Here we have refined our understanding of the non-normalizable modes, relating them as the moduli space parameters which live in a tangent bundle of the moduli space of vacua of the theory.

We have determined the structure of the vortex moduli space, in some cases identifying it with a well-known manifold, and determining the patches needed to cover the whole space. This has been done both restricting to the local (ANO-like) vortices (section 3), and considering the full moduli space (section 4). The latter is closely related to the issue of the sigma model lumps associated to the non-trivial vacuum moduli in these theories [22], as emphasized several times already.

The study of the moduli space of local vortices (section 3) is, on the other hand, deeply related to the nature of non-Abelian *monopoles*: i.e., to the issue of non-Abelian (e.g. GNOW) dualities. Our results in this paper represent further steps along the line of the work [16, 26], even though here we have limited ourselves just to several examples and a few general observations. A more systematic discussion on this problem will be presented elsewhere [51].

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Recently, some non-BPS extensions of $\mathrm{U}(N)$ vortices has been studied for the local case [30] and for the semi-local case [31] with the aim of studying interactions and stability of non-BPS vortices. A non-BPS extension of the $G' = SO, USp$ cases also remains as an open problem. In connection with this, it is known that the $\mathrm{SO}(2M)$ theory admits a non-BPS $\mathbb{Z}_2$ vortex as $\pi_1(\mathrm{SO}(2M) \times \mathrm{U}(1) = \mathbb{Z} \times \mathbb{Z}_2$ [55], which has not been studied in this paper. We limit ourselves to the consideration that such kind of lumps can, in principle, mediate interactions between vortices of opposite chiralities, which, in the range of validity of the moduli space approximation [58], are completely decoupled.

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A The index theorem

We briefly discuss the dimension of the vortex moduli space along the lines of ref. [7], see also refs. [56, 57]. In the following we will keep the gauge group completely generic with a single overall $\mathrm{U}(1)$ factor i.e. $\mathrm{U}(1) \times G'$. Writing the BPS equations ($e = g$) with linear fluctuations $\delta H, \delta\bar{A}$, we obtain

$$\bar{\mathcal{D}}\,\delta H = -i\,\delta\bar{A}\,H\,, \tag{A.1}$$

$$\mathcal{D}\,\delta\bar{A} - \bar{\mathcal{D}}\,\delta A = \frac{ie^2}{2}\mathrm{tr}\left\{\left(\delta H\,H^\dagger + H\,\delta H^\dagger\right)t^\alpha\right\}t^\alpha\,, \tag{A.2}$$

and the Gauss' law reads (with $\nu = 0$)

$$\mathrm{tr}\left[\left(\frac{2}{e^2}\mathcal{D}_\mu F^{\mu\nu} + iH(\mathcal{D}^\nu H)^\dagger - i(\mathcal{D}^\nu H)H^\dagger\right)t^\alpha\right] = 0\,, \quad \forall\alpha\,, \tag{A.3}$$

which we use as a gauge fixing condition [7]

$$\mathcal{D}\,\delta\bar{A} + \bar{\mathcal{D}}\,\delta A = \frac{ie^2}{2}\mathrm{tr}\left\{\left(\delta H\,H^\dagger - H\,\delta H^\dagger\right)t^\alpha\right\}t^\alpha\,. \tag{A.4}$$

A comment in store is that one might wonder why the Gauss law is not already fulfilled by the fact that the solutions to the BPS equations satisfy the Euler-Lagrange equations

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of the system. Fixing the gauge can be done in many different ways, and instead of requiring the fluctuations to be orthogonal to the gauge orbit, it proves convenient to take a direction which corresponds to the time direction of the Gauss law. Even if there is no time dependence of the fields in question, we promote these fluctuations as normal fluctuations rendering the system better manageable. In other words, we constrain the a priori different directions of the fluctuations to obey the linearized Gauss law. This leads to the linear system

$$\bar{\mathcal{D}}\,\delta H = -i\,\delta\bar{A}\,H\,, \tag{A.5}$$

$$\mathcal{D}\,\delta\bar{A} = \frac{ie^2}{2}\mathrm{tr}\left(\delta H\,H^\dagger t^\alpha\right)t^\alpha\;. \tag{A.6}$$

First, we will introduce the following trick

$$\delta\bar{A} = 2\;\mathrm{tr}\left(\delta\bar{A}\;t^\beta\right)t^\beta\,, \tag{A.7}$$

which makes it possible to write the linear system conveniently as the following operator equation

$$\Delta\begin{pmatrix}\delta H\\ \delta\bar{A}\end{pmatrix} = 0\,, \tag{A.8}$$

with (taking $e^2 = 4$ for convenience)

$$\Delta \equiv \begin{pmatrix} i\bar{\mathcal{D}} & -2\;\mathrm{tr}\left(\,\circ\,t^\alpha\right)t^\alpha H\\ 2\;\mathrm{tr}\left(\,\circ\,H^\dagger t^\alpha\right)t^\alpha & i\mathcal{D}\end{pmatrix}\,, \tag{A.9}$$

which has the adjoint operator

$$\Delta^\dagger = \begin{pmatrix} i\mathcal{D} & 2\;\mathrm{tr}\left(\,\circ\,t^\alpha\right)t^\alpha H\\ -2\;\mathrm{tr}\left(\,\circ\,H^\dagger t^\alpha\right)t^\alpha & i\bar{\mathcal{D}}\end{pmatrix}\;. \tag{A.10}$$

Let us start with showing that the operator $\Delta^\dagger$ does not have any zero-modes indeed. That is, the starting point for our vanishing theorem is to take the complex norm $|X|^2 = \mathrm{tr}\,XX^\dagger$ of the operator on a fluctuation

$$\begin{aligned}0 &= \int d^2x\,\left|\Delta^\dagger\begin{pmatrix}X\\ Y\end{pmatrix}\right|^2 \qquad\qquad (\mathrm{A.11})\\ &= \int d^2x\left[|\mathcal{D}X|^2+|\bar{\mathcal{D}}Y|^2+|YH|^2+\left|2\mathrm{tr}\left(XH^\dagger t^\alpha\right)t^\alpha\right|^2+i\mathrm{tr}\partial\left(XH^\dagger Y^\dagger\right)-i\mathrm{tr}\bar{\partial}\left(YHX^\dagger\right)\right],\end{aligned}$$

where the BPS equations have been used together with the fluctuation Y taking part of the algebra $Y = Y^\beta t^\beta$. This forces $Y = 0$. Here we assume the theory to be in the full Higgs phase. We take the fluctuations to vanish at spatial infinity ($|z| \to \infty$), thus the boundary terms can be neglected and we can think of the conditions

$$\bar{\mathcal{D}}X^\dagger = 0\,,\quad \bar{D}Y = 0\,,\quad YH = 0\,,\quad \mathrm{tr}\left(t^\alpha HX^\dagger\right) = 0\,, \tag{A.12}$$

as the BPS equations and F-term conditions for an $\mathcal{N}=2$ $(d=4)$ theory with Y being the adjoint scalar of the vector multiplet and X being anti-chiral fields with the superpotential

$$W = \mathrm{tr}\left(YHX^\dagger\right) . \tag{A.13}$$

Recalling that this toy-theory is evaluated on the background configuration where H is the scalar fields of the vortex and the gauge connection in the covariant derivative $\bar{A}$ is also external fields determined by the background vortex configuration. The vortex configuration can always be rewritten by means of the moduli matrix method yielding $H = S^{-1}H_0(z)$ which gives a holomorphic description of the field $X^\dagger \equiv \tilde{H}$ as $\tilde{H} = \tilde{H}_0 S$ with S the complexified gauge fields of the background configuration. It is now easy to show that the F-term condition yields $\mathrm{tr}\left(t^\alpha H_0(z)\tilde{H}_0(z)\right) = 0$, which in turn simplifies our problem to finding vacuum configurations of this $\mathcal{N}=2$ theory, which has the vacuum in the Higgs phase almost everywhere. We utilize holomorphic invariants $I^i_\mp(H_0, \tilde{H}_0)$ having negative and positive U(1) charges, respectively. The boundary conditions for the invariants are

$$I^i_- = 0 \,, \quad I^i_+ = \mathcal{O}\left(z^{n_i \nu}\right) \,, \tag{A.14}$$

with ν being the U(1) winding. The key point now is to find independent invariants with positive U(1) charges which will reveal the possible existence of a non-zero $\tilde{H}_0$. However, the contrary is important here:

iff there exist no independent I^i_+, then the fluctuations $X^\dagger$ must vanish.

In our cases having $G = \mathrm{U}(1) \times G'$ with $G' = \mathrm{U}(N), \mathrm{SO}(N), \mathrm{USp}(2M)$ with a common U(1) charge for all the fields it is an easy task to show the non-existence of independent holomorphic invariants and the theorem readily applies and completes the proof. We can now go on with the calculation.

Now let us calculate the following two operators $\Delta^\dagger\Delta$ and $\Delta\Delta^\dagger$

$$\Delta^\dagger\Delta = -\mathbf{1}_2\partial\bar{\partial} + \begin{pmatrix} \Gamma_1 + \frac{1}{2}B & L_1 \\ L_2 & \Gamma_2 - \frac{1}{2}B^{\mathrm{adj}} \end{pmatrix} , \tag{A.15}$$

$$\Delta\Delta^\dagger = -\mathbf{1}_2\partial\bar{\partial} + \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix} , \tag{A.16}$$

where $B = F_{12} = -2[\mathcal{D}, \bar{\mathcal{D}}]$ and we have defined the following operators

$$\Gamma_1 X = -iA\bar{\partial}X - i(\bar{\partial}A)X - i\bar{A}\partial X + \bar{A}AX + 2\,\mathrm{tr}\left(XH^\dagger t^\alpha\right)t^\alpha H \,, \tag{A.17}$$

$$\Gamma_2 Y = -i\left[\bar{A}, \partial Y\right] - i\left[\partial\bar{A}, Y\right] - i\left[A, \bar{\partial}Y\right] + \left[A, \left[\bar{A}, Y\right]\right] + 2\,\mathrm{tr}\left(YHH^\dagger t^\alpha\right)t^\alpha \,, \tag{A.18}$$

$$L_1 Y = -iY\mathcal{D}H \,, \tag{A.19}$$

$$L_2 X = i2\,\mathrm{tr}\left(X\bar{\mathcal{D}}H^\dagger t^\alpha\right)t^\alpha \,, \tag{A.20}$$

and the algebra of Y has been used as well as the BPS equations.

To calculate the index of Δ we can evaluate

$$\mathcal{I} = \lim_{M^2\to 0} \mathcal{I}(M^2) = \lim_{M^2\to 0}\left[\mathrm{Tr}\left(\frac{M^2}{\Delta^\dagger\Delta + M^2}\right) - \mathrm{Tr}\left(\frac{M^2}{\Delta\Delta^\dagger + M^2}\right)\right] , \tag{A.21}$$

where Tr denotes a trace over states as well as over the matrices. Now as the eigenvalues of the operator $\Delta^\dagger$ are all positive definite, the index counts only the zero modes of the operator Δ. For well localized solutions (which go to zero faster than $1/r$), the index is independent of M^2. For convenience we can evaluate the index in the limit $M^2 \to \infty$, thus we can expand and obtain

$$\mathcal{I}(M^2) = -M^2\mathrm{Tr}\left[\frac{1}{-\partial\bar{\partial} + M^2}\begin{pmatrix} \frac{1}{2}B & L_1 \\ L_2 & -\frac{1}{2}B^{\mathrm{adj}}\end{pmatrix}\frac{1}{-\partial\bar{\partial} + M^2} + \cdots\right] , \tag{A.22}$$

where the ellipsis denote terms that vanish in taking the limit $M^2 \to \infty$. Tracing over the adjoint field strength gives zero. We can now evaluate the index as

$$\begin{aligned}
\mathcal{I} &= -\lim_{M^2\to\infty} M^2\mathrm{Tr}\int d^2x\, \frac{1}{2}\mathrm{tr}(F_{12})\left\langle x\left|\left(-\partial\bar{\partial} + M^2\right)^{-2}\right|x\right\rangle , \\
&= -\lim_{M^2\to\infty} M^2 \sum_1^{N_\mathrm{F}}\int d^2x\, \frac{N}{2\sqrt{2N}}F_{12}^0 \int \frac{d^2k}{(2\pi)^2}\frac{1}{\left(\frac{1}{4}k^2 + M^2\right)^2} , \\
&= N_\mathrm{F} N\nu ,
\end{aligned} \tag{A.23}$$

where

$$\nu = -\frac{1}{2\pi\sqrt{2N}}\int d^2x\, F_{12}^0 = \frac{k}{n_0} . \tag{A.24}$$

Because of the vanishing theorem, the index gives exactly the number of (complex) zero-modes for the BPS equations for the vortex. Thus we obtain the same number of zero-modes as the number of moduli parameters in the moduli matrix formalism. Note that the result is obtained independently of the gauge group (however only valid when the vanishing theorem applies) and the impact of the group is simply encoded in ν. We also note that our result reduces to that of ref. [7] for $\mathrm{U}(N)$ by recalling that $\nu = k/N$ in that case.

B The orientation vectors

We have considered the moduli matrix per se and studied the orientational moduli space of the local non-Abelian vortices. Our result for $G' = \mathrm{SO}(2M), \mathrm{USp}(2M)$ is the quotient space given in eq. (3.6). These spaces are well-known Hermitian symmetric spaces [48, 49]. They can be embedded in the complex Grassmann space $Gr_{2M,M} \simeq \mathrm{SU}(2M)/[\mathrm{SU}(M)\times\mathrm{SU}(M)\times\mathrm{U}(1)]$ which is described by a $2M\times M$ complex matrix via a $\mathrm{GL}(M,\mathbb{C})$ equivalence relation

$$Gr_{2M,M} \simeq \Phi /\!/ \mathrm{GL}(M,\mathbb{C}) = \{\Phi \sim \Phi\mathcal{V}\} , \quad \mathcal{V} \in \mathrm{GL}(M,\mathbb{C}) . \tag{B.1}$$

where the action of $\mathrm{GL}(M,\mathbb{C})$ is free. In other words we require the rank of Φ to be M. The embedding is defined by the constraint [49]

$$\Phi^\mathrm{T} J\Phi = 0 , \tag{B.2}$$

where J is given by eq. (2.23).

We can relate the matrix Φ to the orientation of the local vortex as follows. Notice that the moduli matrix decreases its rank by M at the "vortex center", $z = z_0$. The orientational moduli can be extracted as M linearly independent $2M$-vectors orthogonal to $H_0(z = z_0)$ [10, 11]

$$H_0(z = z_0)\, \vec{\phi}_i = 0\,, \qquad (i = 1, 2, \ldots, M)\ . \tag{B.3}$$

Let us thus define a $2M \times M$ orientational matrix by putting $\vec{\phi}_i$ $(i = 1, 2, \ldots)$ all together as

$$\Phi = \left(\vec{\phi}_1, \vec{\phi}_2, \ldots, \vec{\phi}_M\right)\,, \qquad H_0(z = z_0)\, \Phi = 0\ . \tag{B.4}$$

As Φ' given by $\Phi' \equiv \Phi\mathcal{V}$ with $\mathcal{V} \in \mathrm{GL}(M, \mathbb{C})$ — which is just a change of the basis — satisfies the same equation (B.3), Φ' represents the same physical configuration as Φ. This leads to the equivalence relation (B.1) and to the complex Grassmannian $Gr_{2M,M}$, as claimed. The isotropic condition (B.2) can be found as follows. The strong condition (3.1) is written as

$$(H_0\Phi)^{\mathrm{T}} J (H_0\Phi) = z\Phi^{\mathrm{T}} J\Phi\ . \tag{B.5}$$

Taking the derivative of this with respect to z, one obtains

$$(\partial H_0\Phi)^{\mathrm{T}} J H_0\Phi + (H_0\Phi)^{\mathrm{T}} J \partial H_0\Phi = \Phi^{\mathrm{T}} J\Phi\ . \tag{B.6}$$

Evaluating this at $z = z_0$ one is led to the constraint (B.2).

The advantage of considering Φ instead of $H_0(z)$ is simplification of the calculation. In the rest of this subsection, one can completely forget the previous argument of the moduli matrix. All the results derived from H_0 can be reproduced by Φ alone. Let us explain this by taking two examples: SO(4) and USp(4). Then Φ is a 4×2 matrix satisfying $\Phi^{\mathrm{T}} J\Phi = 0$. Since Φ has rank 2, we can generally bring Φ onto the following form by using $\mathrm{GL}(2, \mathbb{C})$

$$\Phi^{(\frac{1}{2},\frac{1}{2})}_{\mathrm{SO}(4)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -b \\ b & 0 \end{pmatrix}\,, \qquad \Phi^{(\frac{1}{2},\frac{1}{2})}_{\mathrm{USp}(4)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ b & c \end{pmatrix}\ . \tag{B.7}$$

Of course, further three patches $\{\Phi^{(-\frac{1}{2},\frac{1}{2})}, \Phi^{(\frac{1}{2},-\frac{1}{2})},\ \Phi^{(-\frac{1}{2},-\frac{1}{2})}\}$ are obtained by fixing $\mathrm{GL}(2, \mathbb{C})$ in such a way that the {2-3 rows, 1-4 rows, 3-4 rows} become the unit matrix, respectively.

The transition functions among them are given through the $\mathrm{GL}(2, \mathbb{C})$. In the case of $G' = \mathrm{USp}(4)$, the transition functions from the $(\frac{1}{2}, \frac{1}{2})$-patch to the $\{(-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})\}$-patches are given by

$$\mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},\frac{1}{2})}_{\mathrm{USp}(4)} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}^{-1}\,, \quad \mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(\frac{1}{2},-\frac{1}{2})}_{\mathrm{USp}(4)} = \begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix}^{-1}\,, \quad \mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},-\frac{1}{2})}_{\mathrm{USp}(4)} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1}\ . \tag{B.8}$$

When the inverse of $\mathcal{V}$ does not exist, such points are not covered by two patches but only by one of them. In the case of $G' = \mathrm{SO}(2M)$, neither $\mathcal{V}^{(-\frac{1}{2},\frac{1}{2})\to(\frac{1}{2},\frac{1}{2})}$ nor $\mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},\frac{1}{2})}$ have an inverse. Thus the $(\frac{1}{2},\frac{1}{2})$-patch is disconnected from the $(-\frac{1}{2},\frac{1}{2})$-patch and the $(\frac{1}{2},-\frac{1}{2})$-patch. It connects only with the $(-\frac{1}{2},-\frac{1}{2})$-patch and the transition function is given by

$$\mathcal{V}_{\mathrm{SO}(4)}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},-\frac{1}{2})} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}^{-1} . \tag{B.9}$$

Similarly, the $(-\frac{1}{2},\frac{1}{2})$-patch and the $(\frac{1}{2},-\frac{1}{2})$-patch are connected. This is a reinterpretation of the $\mathbb{Z}_2$-parity of the local vortex in the model with $G' = \mathrm{SO}(4)$, see figure 2. An extension of this to the local vortex in $G' = \mathrm{SO}(2M)$ is straightforward.

C Some details

C.1 Spatially-separated vortices

When the two vortices are separated, i.e. $\delta \neq 0$, the second equation of eq. (3.33) (together with $\mathrm{Tr}\,\Gamma = 0$) is solved by

$$\Gamma = o'\,\Gamma_0\,o'^{-1}, \qquad \Gamma_0 \equiv \sqrt{\delta}\begin{pmatrix} \mathbf{1}_{M-r} & \\ & -\mathbf{1}_{M-r} \end{pmatrix}. \tag{C.1}$$

There remains an arbitrariness under reshuffling the form,

$$o' \to o's, \quad \Gamma_0 \to s^{-1}\Gamma_0 s, \qquad s \equiv \begin{pmatrix} u_1' & \\ & u_2' \end{pmatrix}, \tag{C.2}$$

where $u_i' \in \mathrm{GL}(M-r,\mathbb{C})$. Then the first condition in eq. (3.33) leads to

$$o'^{\mathrm{T}} J_{2(M-r)}\, o' = \begin{pmatrix} 0 & X \\ \epsilon\, X^{\mathrm{T}} & 0 \end{pmatrix} \sim J_{2(M-r)}\,, \tag{C.3}$$

where we have used the above-mentioned freedom to arrive at the last form for $J_{2(M-r)}$. The above relation means that o' is an element of $\mathrm{O}(2(M-r))^{\mathbb{C}}$ $(\mathrm{USp}(2(M-r))^{\mathbb{C}})$. There exists still an unphysical transformation $u_1'^{\mathrm{T}} = u_2'^{-1} \equiv u \in \mathrm{GL}(M-r,\mathbb{C})$. Thus the solution of the strong condition (3.33) with $\delta \neq 0$ is given by

$$\Gamma \in \begin{cases} \left\{\mathbb{C}^* \times \left[\frac{\mathrm{O}(2(M-r),\mathbb{C})}{\mathrm{U}(M-r)}\right]^{\mathbb{C}}\right\}/\mathbb{Z}_2 & \text{for} \quad G' = \mathrm{SO}(2M)\,, \\ \left\{\mathbb{C}^* \times \left[\frac{\mathrm{USp}(2(M-r),\mathbb{C})}{\mathrm{U}(M-r)}\right]^{\mathbb{C}}\right\}/\mathbb{Z}_2 & \text{for} \quad G' = \mathrm{USp}(2M)\,, \end{cases} \tag{C.4}$$

with the first $\mathbb{C}^*$ factor being the relative distance $\sqrt{\delta}$. The $\mathbb{Z}_2$ factors in the denominators come about due to the fact that a combination of a π-rotation in the x^1-x^2 space $\sqrt{\delta} \to -\sqrt{\delta}$ and a permutation $o' \to o'p$, satisfying $p\Gamma_0 p^{-1} = -\Gamma_0$ is an identity operation.

C.2 Fixing NG modes for section 3.2.1

Let us go into a detailed investigation, in order to verify the results in section 3.2.1. In the first place note that $a_{0;A,S}$ and $C_{1,2}$ are obviously NG modes when two vortices are coincident, namely $\delta = 0$. One can confirm this fact, for example, by considering an infinitesimal color-flavor G'_{C+F} transformation accompanied by an appropriate V-transformation. Therefore any moduli matrices of the form (3.28) can be always brought into the following

$$H_0^{(\overbrace{1,\ldots,1}^{r},\overbrace{0,\ldots,0}^{M-r})} = \begin{pmatrix} (z-z_0)^2\mathbf{1}_r & 0 & 0 & 0 \\ 0 & (z-z_0)\mathbf{1}_{M-r}+\Gamma_{11} & 0 & \Gamma_{12} \\ a_{1;A,S}\,z & 0 & \mathbf{1}_r & 0 \\ 0 & \Gamma_{21} & 0 & (z-z_0)\mathbf{1}_{M-r}+\Gamma_{22} \end{pmatrix} . \tag{C.5}$$

For $\delta = 0$, the rank $2\gamma = \mathrm{rank}(\Gamma)$ is less than $2\gamma < 2(M-r)$. The first condition in eq. (3.33) states that $\Gamma J_{2(M-r)}$ is anti-symmetric (symmetric), so that Γ can be written as

$$\Gamma = \epsilon\, q \tilde{J}_{2\gamma}\, q^{\mathrm{T}}\, J_{2(M-r)}\,, \tag{C.6}$$

where q is a $2(M-r)\times 2\gamma$ matrix whose rank is 2γ, $(M-r\geq\gamma)$, and $\tilde{J}_{2\gamma}$ is the invariant tensor of $\tilde{G}'_{2\gamma} = \mathrm{USp}(2\gamma)$ for $G' = \mathrm{SO}(2M)$ and $\tilde{G}'_{2\gamma} = \mathrm{SO}(2\gamma)$ for $G' = \mathrm{USp}(2M)$. Then the second condition is translated into the following constraint on q:

$$A = 0\,, \quad A \equiv q^{\mathrm{T}} J_{2(M-r)} q\;. \tag{C.7}$$

Note that the rank of $A = q^{\mathrm{T}} J_{2(M-r)} q$ is bounded as

$$4\gamma - 2(M-r) \leq \mathrm{rank}(A) \leq \mathrm{rank}(q) = 2\gamma\;. \tag{C.8}$$

Therefore, $2\gamma \leq M-r$ in the present case of $\mathrm{rank}(A) = 0$. This last condition can be solved by

$$q = O\begin{pmatrix} g \\ \mathbf{0}_{2(M-r-\gamma)\times 2\gamma}\end{pmatrix}, \quad g\in \mathrm{GL}(2\gamma,\mathbb{C})\,, \quad O \in G'_{2(M-r)}\;. \tag{C.9}$$

Thus we find

$$\Gamma = O\left(\begin{array}{c|c} \begin{matrix} g\tilde{J}_{2\gamma}g^{\mathrm{T}} & \\ & \mathbf{0}_{M-r-2\gamma}\end{matrix} & \\ \hline & \begin{matrix} \mathbf{0}_{2\gamma} & \\ & \mathbf{0}_{M-r-2\gamma}\end{matrix}\end{array}\right) O^{\mathrm{T}} J_{2(M-r)}\;. \tag{C.10}$$

In the case of $G' = \mathrm{SO}(2M)$, we can bring the anti-symmetric matrix $g\tilde{J}_{2\gamma}g^{\mathrm{T}}$ onto a block-diagonal form as

$$g\tilde{J}_{2\gamma}g^{\mathrm{T}} = u\,\Lambda\,u^{\mathrm{T}}, \quad \Lambda \equiv i\sigma_2\otimes \mathrm{diag}(\lambda_1\mathbf{1}_{p_1},\lambda_2\mathbf{1}_{p_2},\ldots,\lambda_q\mathbf{1}_{p_q}), \quad (\lambda_i > \lambda_{i+1} > 0), \tag{C.11}$$

where $u\in \mathrm{U}(2\gamma)$ and $2\sum_{i=1}^{q} p_i = 2\gamma$. Thus we have found

$$\Gamma = O'\left(\begin{array}{c|c} & \begin{matrix} \Lambda & \\ & \mathbf{0}_{M-r-2\gamma}\end{matrix} \\ \hline \begin{matrix} \mathbf{0}_{2\gamma} & \\ & \mathbf{0}_{M-r-2\gamma}\end{matrix} & \end{array}\right) O'^{-1}\,, \tag{C.12}$$

$$
O' \equiv O \begin{pmatrix} u & & & \\ & \mathbf{1}_{M-r-2\gamma} & & \\ & & \left(u^{\mathrm{T}}\right)^{-1} & \\ & & & \mathbf{1}_{M-r-2\gamma} \end{pmatrix} \in \mathrm{SO}(2(M-r))\ . \tag{C.13}
$$

Similarly, the anti-symmetric tensor $a_{1;,A}$ can be brought onto a diagonal form. Let $\operatorname{rank}(a_{1,A}) = 2\alpha \leq r$, then we obtain

$$
a_{1;A} = \begin{pmatrix} \mathbf{0}_{r-\alpha} & \\ & u'\,\Lambda'\,u'^{\mathrm{T}} \end{pmatrix}\ , \qquad \Lambda' \equiv i\sigma_2 \otimes \operatorname{diag}(\lambda'_1 \mathbf{1}_{p'_1}, \lambda'_2 \mathbf{1}_{p'_2}, \ldots, \lambda'_{q'} \mathbf{1}_{p'_{q'}})\,, \tag{C.14}
$$

where $u' \in \mathrm{U}(2\alpha)$, $2\sum_{i=1}^{q'} p'_i = 2\alpha$ and $\lambda'_i > \lambda'_{i+1} > 0$. Finally, we arrive at the following expression

$$
H_0 = \left(\begin{array}{cc|cc||cc|cc}
z^2\mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z^2\mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & z\mathbf{1}_{2\gamma} & 0 & 0 & 0 & \Lambda & 0 \\
0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & \mathbf{0}_{M-r-2\gamma} \\
\hline\hline
\mathbf{0}_{r-2\alpha} & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 \\
0 & \Lambda'\,z & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{2\gamma} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma}
\end{array}\right)\ , \tag{C.15}
$$

where we have turned off the center of mass $z_0 = 0$. One can return to the previous moduli matrix by using the color-flavor symmetry $H_0 \to U^{-1} H_0 U$ with

$$
U \equiv \left(\begin{array}{cc|c||cc|c}
\mathbf{1}_{r-2\alpha} & & & & & \\
& u'^{\mathrm{T}} & & & & \\
\hline
& & O'^{-1} & & & \\
\hline\hline
& & & \mathbf{1}_{r-2\alpha} & & \\
& & & & u'^{-1} & \\
\hline
& & & & & O'^{-1}
\end{array}\right) \in \mathrm{SO}(2M)\ . \tag{C.16}
$$

By making use of the V-transformation, one can bring this onto the following form

$$
VH_0 = \left(\begin{array}{cc|cc||cc|cc}
z^2\mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z^2\mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & z^2\mathbf{1}_{2\gamma} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 \\
0 & \Lambda'\,z & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 \\
\hline
0 & 0 & \Lambda^{-1} z & 0 & 0 & 0 & \mathbf{1}_{2\gamma} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma}
\end{array}\right)\ , \tag{C.17}
$$

$$V=\left(\begin{array}{cc|cc||cc|cc}
\mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & z\mathbf{1}_{2\gamma} & 0 & 0 & 0 & -\Lambda & 0 \\
0 & 0 & 0 & \mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 \\
\hline
0 & 0 & \Lambda^{-1} & 0 & 0 & 0 & \mathbf{0}_{2\gamma} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma}
\end{array}\right). \tag{C.18}$$

where one can check that $V \in \mathrm{SO}(2M,\mathbb{C})$ because $\Lambda^{\mathrm{T}} = -\Lambda$. We can rearrange the eigenvalues $\tilde{\lambda}_a = \{\lambda_i^{-1}, \lambda_j'\}$ in such a way that

$$\mathrm{diag}\left(\Lambda',\ \Lambda^{-1}\right) = i\sigma_2 \otimes \mathrm{diag}\left(\tilde{\lambda}_1 \mathbf{1}_{\tilde{p}_1}, \ldots, \tilde{\lambda}_s \mathbf{1}_{\tilde{p}_s}\right), \quad \tilde{\lambda}_a > \tilde{\lambda}_{a+1} > 0\,, \tag{C.19}$$

hence the $G'_{\mathrm{C+F}} = \mathrm{SO}(2M)$ orbit can easily be seen in eq. (3.56).

The arguments for $G' = \mathrm{USp}(2M)$ are analogous to those of $G' = \mathrm{SO}(2M)$. A small difference is that $J_{2(M-r)}\Gamma$ and $a_{1;S}$ are now symmetric. In the end, we obtain the moduli matrix on the following form

$$H_0 = \left(\begin{array}{ccc|ccc}
z^2\mathbf{1}_{r-\beta} & 0 & 0 & 0 & 0 & 0 \\
0 & z^2\mathbf{1}_{\beta+\zeta} & 0 & 0 & 0 & 0 \\
0 & 0 & z\mathbf{1}_{M-r-\zeta} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \mathbf{1}_{r-\beta} & 0 & 0 \\
0 & \tilde{\Lambda}\, z & 0 & 0 & \mathbf{1}_{\beta+\zeta} & 0 \\
0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-\zeta}
\end{array}\right), \tag{C.20}$$

$$\tilde{\Lambda} = \mathrm{diag}(\tilde{\lambda}_1 \mathbf{1}_{\tilde{p}_1}, \ldots, \tilde{\lambda}_s \mathbf{1}_{\tilde{p}_s})\,, \tag{C.21}$$

with $\beta = \mathrm{rank}(\Gamma)$ and $\zeta = \mathrm{rank}(a_{1;S})$.

D Some transition functions

Here we make a collection of some of the transition functions discussed in the main text.

D.1 Example 1

The transition functions between two $\mathbb{Z}_2$-parity $+1$ patches for the minimal semi-local vortices in $G' = \mathrm{SO}(4)$ theory of section 4.2.1:

$$\begin{cases}
a = -f'i' + \frac{a'+d'}{2}\,, \\
b = -g'i'\,, \\
c = e'i'\,, \\
d = f'i' + \frac{a'+d'}{2}\,, \\
e = -c'i'\,, \\
f = \frac{(a'-d')i'}{2}\,, \\
g = b'i'\,, \\
i = -\frac{1}{i'}\,.
\end{cases} \tag{D.1}$$

D.2 Example 2

The transition functions between $H_0^{(0,0)}$ and $H_0^{(1,1)}$ for the $k=2$ semi-local vortices in $G' = \mathrm{SO}(4)$ theory of section 4.3.1:

$$\begin{cases} a_0 = \frac{1}{2}a'_1 - \frac{1}{2}d'_1 + \frac{i'_0}{i'_1}\,, \\ b_0 = b'_1\,, \\ c_0 = e'_1\,, \\ d_0 = f'_1 - \frac{1}{i'_1}\,, \\ e_0 = c'_1\,, \\ f_0 = -\frac{1}{2}a'_1 + \frac{1}{2}d'_1 + \frac{i'_0}{i'_1}\,, \\ g_0 = f'_1 + \frac{1}{i'_1}\,, \\ h_0 = g'_1\,, \\ i_0 = -c'_1 i'_0 - c'_0 i'_1 + \frac{1}{2}a'_1 c'_1 i'_1 + \frac{1}{2}c'_1 d'_1 i'_1\,, \\ j_0 = a'_1 i'_0 - \frac{i'^2_0}{i'_1} - \frac{1}{4}a'^2_1 i'_1 - d'_0 i'_1 + \frac{1}{4}d'^2_1 i'_1\,, \\ k_0 = \frac{1}{2}a'_1 + \frac{1}{2}d'_1 - f'_1 i'_0 - f'_0 i'_1 - \frac{i'_0}{i'_1} + \frac{1}{2}a'_1 f'_1 i'_1 + \frac{1}{2}d'_1 f'_1 i'_1\,, \\ l_0 = -g'_1 i'_0 - g'_0 i'_1 + \frac{1}{2}a'_1 g'_1 i'_1 + \frac{1}{2}d'_1 g'_1 i'_1\,, \\ m_0 = -d'_1 i'_0 + a'_0 i'_1 + \frac{i'^2_0}{i'_1} - \frac{1}{4}a'^2_1 i'_1 + \frac{1}{4}d'^2_1 i'_1\,, \\ n_0 = b'_1 i'_0 + b'_0 i'_1 - \frac{1}{2}a'_1 b'_1 i'_1 - \frac{1}{2}d'_1 b'_1 i'_1\,, \\ o_0 = e'_1 i'_0 + e'_0 i'_1 - \frac{1}{2}a'_1 e'_1 i'_1 - \frac{1}{2}d'_1 e'_1 i'_1\,, \\ p_0 = \frac{1}{2}a'_1 + \frac{1}{2}d'_1 + f'_1 i'_0 + f'_0 i'_1 - \frac{i'_0}{i'_1} - \frac{1}{2}a'_1 f'_1 i'_1 - \frac{1}{2}d'_1 f'_1 i'_1. \end{cases} \tag{D.2}$$

These transition functions are, of course, invertible.

D.3 Example 3

The transition functions between the patches with $\mathbb{Z}_2$-parity -1, viz. $H_0^{(1,0)}$ and $H_0^{(-1,0)}$, for the $k=2$ semi-local vortices in $G' = \mathrm{SO}(4)$ theory discussed in section 4.3.1, are

$$\begin{cases} a_1 = -c'_1 e'_1 i'_1 - \frac{e'_0}{e'_1} - \frac{i'_0}{i'_1}\,, \\ a_0 = -c'_0 e'_1 i'_1 + \frac{e'_0 i'_0}{e'_1 i'_1}\,, \\ b_1 = -b'_1 e'_1 i'_1 - f'_0 i'_1 - e'_1 j'_0 + \frac{e'_0 i'_1}{e'_1}\,, \\ b_0 = -b'_0 e'_1 i'_1 - f'_0 i'_0 - e'_0 j'_0 + \frac{e'_0 i'_0}{e'_1}\,, \\ c_1 = -a'_1 e'_1 i'_1 - e'_0 i'_1 - e'_1 i'_0\,, \\ c_0 = -a'_0 e'_1 i'_1 + e'_0 i'_0\,, \\ d_1 = -d'_1 e'_1 i'_1 - g'_0 i'_1 - e'_1 k'_0 - \frac{e'_1 i'_0}{i'_1}\,, \\ d_0 = -d'_0 e'_1 i'_1 + g'_0 i'_0 + e'_0 k'_0 - \frac{e'_0 i'_0}{i'_1} \\ e_1 = -\frac{1}{i'_1}\,, \\ e_0 = -\frac{i'_0}{i'^2_1}\,, \\ f_0 = -\frac{i'_0}{i'_1} - \frac{e'_1 j'_0}{i'_1}\,, \\ g_0 = -\frac{e'_1 k'_0}{i'_1} - \frac{e'_1 i'_0}{i'^2_1}\,, \\ i_1 = -\frac{1}{e'_1}\,, \\ i_0 = -\frac{e'_0}{e'^2_1}\,, \\ j_0 = -\frac{i'_1 f'_0}{e'_1} - \frac{i'_1 e'_0}{e'^2_1}\,, \\ k_0 = -\frac{g'_0 i'_1}{e'_1} - \frac{e'_0}{e'_1}\,. \end{cases} \tag{D.3}$$

D.4 Example 4

The transition functions between the patches (-1) and (1) for the $k=1$ semi-local vortices in $G' = \mathrm{SO}(3)$ theory discussed in section 4.4.1, are

$$\begin{cases} d \;= -\frac{2}{d'} \,, \\ e \;= -\frac{2e'}{d'^2} \,, \\ z_3 = -\frac{2e'}{d'} - z_3' \,, \\ f \;= d'e' - \frac{1}{2} d'^2 z_1' \,, \\ a \;= \frac{1}{2}\left(e'^2 - d'^2 z_2'\right) \,, \\ b \;= -\frac{1}{2} b' d'^2 - e' - d' z_3' \,, \\ c \;= -\frac{1}{2} c' d'^2 - e' \left(\frac{e'}{d'} + z_3'\right) \,, \\ z_1 = \frac{2e'}{d'} - \frac{1}{2} d'^2 f' \,, \\ z_2 = \frac{e'^2}{d'^2} - \frac{1}{2} a' d'^2 \,. \end{cases} \tag{D.4}$$

D.5 Example 5

The transition functions between the patches $(1,1)$ and $(0,0)$ for the $k=1$ semi-local vortices in $G' = \mathrm{SO}(5)$ theory discussed in section 4.4.2, are

$$\begin{cases} a_1' = \frac{a_1 - a_4}{2} + \frac{f}{e} + \frac{i_1 i_2}{2e} \,, \\ a_2' = a_2 + \frac{i_2^2}{2e} \,, \\ a_3' = c_1 \,, \\ a_4' = -\frac{1}{e} + c_2 \,, \\ a_5' = g_1 - \frac{i_2}{e} \,, \\ b_1' = a_3 - \frac{i_1^2}{2e} \,, \\ b_2' = -\frac{a_1 - a_4}{2} + \frac{f}{e} - \frac{i_1 i_2}{2e} \,, \\ b_3' = \frac{1}{e} + c_2 \,, \\ b_4' = c_3 \,, \\ b_5' = g_2 + \frac{i_1}{e} \,, \\ c_1' = -e b_3 + \frac{e a_3 (a_1 + a_4)}{2} - a_3 f - \frac{i_1 (a_1 i_1 + a_3 i_2)}{2} - i_1 j_1 \,, \\ c_2' = -e b_4 - \frac{e\left(a_1^2 - a_4^2\right)}{4} + a_1 f - \frac{f^2}{e} - \frac{i_1 (a_2 i_1 + a_4 i_2)}{2} - i_1 j_2 \,, \\ c_3' = -e d_2 + \frac{c_2 e (a_1 + a_4)}{2} + \frac{a_1 + a_4}{2} - \frac{f}{e} - c_2 f - \frac{i_1 (c_1 i_1 + c_2 i_2)}{2} - \frac{i_1 i_2}{2e} \,, \\ c_4' = -e d_3 + \frac{c_3 e (a_1 + a_4)}{2} - c_3 f + \frac{i_1^2}{2e} - \frac{i_1 (c_2 i_1 + c_3 i_2)}{2} \,, \\ c_5' = -e h_2 + \frac{g_2 e (a_1 + a_4)}{2} - f g_2 + \frac{i_1 (a_1 + a_4)}{2} - \frac{f i_1}{e} - \frac{i_1 (g_1 i_1 + g_2 i_2)}{2} - i_1 y \,, \\ d_1' = e b_1 - \frac{e\left(a_1^2 - a_4^2\right)}{4} - a_4 f + \frac{f^2}{e} - \frac{i_2 (a_1 i_1 + a_3 i_2)}{2} - i_2 j_1 \,, \\ d_2' = e b_2 - \frac{a_2 e (a_1 + a_4)}{2} + a_2 f - \frac{i_2 (a_2 i_1 + a_4 i_2)}{2} - i_2 j_2 \,, \\ d_3' = e d_1 - \frac{c_1 e (a_1 + a_4)}{2} + c_1 f - \frac{i_2 (c_1 i_1 + c_2 i_2)}{2} - \frac{i_2^2}{2e} \,, \\ d_4' = e d_2 - \frac{c_2 e (a_1 + a_4)}{2} + \frac{a_1 + a_4}{2} - \frac{f}{e} + c_2 f - \frac{i_2 (c_2 i_1 + c_3 i_2)}{2} + \frac{i_1 i_2}{2e} \,, \\ d_5' = e h_1 - \frac{g_1 e (a_1 + a_4)}{2} + f g_1 + \frac{i_2 (a_1 + a_4)}{2} - \frac{f i_2}{e} - \frac{i_2 (g_1 i_1 + g_2 i_2)}{2} - i_2 y \,, \\ e_1' = j_1 + \frac{i_1 (a_1 - a_4)}{2} + \frac{f i_1}{e} + a_3 i_2 \,, \\ e_2' = j_2 - \frac{i_2 (a_1 - a_4)}{2} + \frac{f i_2}{e} + a_2 i_1 \,, \\ e_3' = c_1 i_1 + c_2 i_2 + \frac{i_2}{e} \,, \\ e_4' = c_2 i_1 + c_3 i_2 - \frac{i_1}{e} \,, \\ e_5' = y + g_1 i_1 + g_2 i_2 \,, \end{cases} \tag{D.5}$$

D.6 Example 6

The transition functions between the patches $(-1,0)$ and $(1,0)$ for the $k=1$ semi-local vortices in $G'=\mathrm{SO}(5)$ theory discussed in section 4.4.2, are

$$
\left\{
\begin{aligned}
a_0 &= \tfrac{2f'_0f'_1+h'^2_0}{\Xi}-\tfrac{1}{2}b'_0\Xi\,,\\
a_1 &= -\tfrac{2\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi}-\tfrac{1}{2}b'_1\Xi\,,\\
b_0 &= f'_0f'_1+\tfrac{1}{2}h'^2_0-\tfrac{1}{2}a'_0\Xi\,,\\
b_1 &= -e'_0f'_0-e'_1f'_1-h'_0j'_1-\tfrac{1}{2}a'_1\Xi\,,\\
c_0 &= f'_1g'_0+f'_0g'_2+h'_0h'_1+\tfrac{e'_0\left(2f'_0f'_1+h'^2_0\right)}{\Xi}-\tfrac{1}{2}c'_0\Xi\,,\\
c_1 &= f'_1-e'_0g'_0-e'_1g'_2-h'_1j'_1-\tfrac{2e'_0\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi}-\tfrac{1}{2}c'_1\Xi\,,\\
d_0 &= f'_1g'_1+f'_0g'_3+h'_0h'_2+\tfrac{e'_1\left(2f'_0f'_1+h'^2_0\right)}{\Xi}-\tfrac{1}{2}d'_0\Xi\,,\\
d_1 &= f'_0-e'_0g'_1-e'_1g'_3-h'_2j'_1-\tfrac{2e'_1\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi}-\tfrac{1}{2}d'_1\Xi\,,\\
e_0 &= -\tfrac{2e'_0}{\Xi}\,,\\
e_1 &= -\tfrac{2e'_1}{\Xi}\,,\\
f_0 &= \tfrac{-4e'^2_1f'_1-4e'_1h'_0j'_1+2f'_0j'^2_1}{\Xi^2}\,,\\
f_1 &= \tfrac{-4e'^2_0f'_0-4e'_0h'_0j'_1+2f'_1j'^2_1}{\Xi^2}\,,\\
g_0 &= g'_0-\tfrac{2e'_1\left(e'_0g'_0+e'_1g'_2+h'_1j'_1\right)}{\Xi}+\tfrac{2e'_0\left(-2e'^2_1f'_1-2e'_1h'_0j'_1+f'_0j'^2_1\right)}{\Xi^2}\,,\\
g_1 &= \tfrac{4e'^3_1\left(f'_1+e'_0g'_3\right)+g'_1j'^4_1-2e'^2_1j'_1\left(2h'_0+2e'_0h'_2+g'_3j'_1\right)+2e'_1j'^2_1\left(f'_0+e'_0g'_1-h'_2j'_1\right)}{\Xi^2}\,,\\
g_2 &= \tfrac{-4e'^3_0\left(f'_0+e'_1g'_0\right)+g'_2j'^4_1-2e'^2_0j'_1\left(2h'_0+2e'_1h'_1+g'_0j'_1\right)+2e'_0j'^2_1\left(f'_1+e'_1g'_2-h'_1j'_1\right)}{\Xi^2}\,,\\
g_3 &= g'_3-\tfrac{4e'_0e'_1\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi^2}-\tfrac{2\left(-e'_1f'_1+e'^2_0g'_1+e'_0e'_1g'_3+e'_0h'_2j'_1\right)}{\Xi}\,,\\
h_0 &= \tfrac{-4e'_0e'_1h'_0+4e'_0f'_0j'_1+4e'_1f'_1j'_1+2h'_0j'^2_1}{\Xi^2}\,,\\
h_1 &= \tfrac{4e'^2_0\left(-e'_1\left(h'_0+e'_1h'_1\right)+\left(f'_0+e'_1g'_0\right)j'_1\right)+j'^3_1\left(2e'_1g'_2+h'_1j'_1\right)+2e'_0j'_1\left(2e'_1\left(f'_1+e'_1g'_2\right)+j'_1\left(h'_0+g'_0j'_1\right)\right)}{\Xi^2}\,,\\
h_2 &= \tfrac{-4e'_0e'^2_1\left(h'_0+e'_0h'_2\right)+4e'_1\left(e'_1f'_1+e'_0\left(f'_0+e'_0g'_1+e'_1g'_3\right)\right)j'_1+2e'_1h'_0j'^2_1+2\left(e'_0g'_1+e'_1g'_3\right)j'^3_1+h'_2j'^4_1}{\Xi^2}\,,\\
i_0 &= \tfrac{2e'^2_0e'^2_1i'_0-2e'_0e'_1\left(f'_1j'_0-i'_0j'^2_1+f'_0j'_2+h'_0k'\right)+j'_1\left(-f'_1j'_0j'_1+\frac{1}{2}i'_0j'^3_1-f'_0\left(2f'_1+j'_1j'_2\right)-h'_0\left(h'_0+j'_1k'\right)\right)}{\Xi}\,,\\
i_1 &= \tfrac{2e'^2_0e'_1\left(e'_1i'_1+j'_0\right)+j'_1\left(2e'_1f'_1+h'_0j'_1+\frac{1}{2}i'_1j'^3_1+e'_1j'_1j'_2+j'^2_1k'\right)+e'_0\left(j'_1\left(2f'_0+j'_0j'_1\right)+2e'^2_1j'_2+2e'_1\left(-h'_0+j'_1\left(i'_1j'_1+k'\right)\right)\right)}{\Xi}\,,\\
j_0 &= \tfrac{-j'^3_1\left(2f'_0+j'_0j'_1\right)+4e'_0e'^3_1j'_2+2e'^2_1j'_1\left(2f'_1+j'_1j'_2+2e'_0k'\right)+2e'_1j'^2_1\left(2h'_0-e'_0j'_0+j'_1k'\right)}{\Xi^2}\,,\\
j_1 &= \tfrac{2j'_1}{\Xi}\,,\\
j_2 &= \tfrac{4e'^3_0e'_1j'_0-j'^3_1\left(2f'_1+j'_1j'_2\right)+2e'^2_0j'_1\left(2f'_0+j'_0j'_1+2e'_1k'\right)+2e'_0j'^2_1\left(2h'_0-e'_1j'_2+j'_1k'\right)}{\Xi^2}\,,\\
k &= \tfrac{-2e'_0j'_1\left(-2e'_1h'_0+j'_1\left(2f'_0+j'_0j'_1\right)+2e'^2_1j'_2\right)+4e'^2_0e'_1\left(-j'_0j'_1+e'_1k'\right)-j'^2_1\left(4e'_1f'_1+2h'_0j'_1+2e'_1j'_1j'_2+j'^2_1k'\right)}{\Xi^2}\,.
\end{aligned}
\right.
\tag{D.6}
$$

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